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Dominating Coloring Number of Claw-free Graphs

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Abstract

Let G be a graph. It is well-known that G contains a proper vertex-coloring with $\chi(G)$ colors with the property that at least one color class of the coloring is a dominating set in G. Among all such proper vertex-coloring of the vertices of G, a coloring with the maximum number of color classes that are dominating sets in G is called a dominating-χ-coloring of G. The number of color classes that are dominating sets in a dominating-χ-coloring of G is defined to be the dominating-χ-color number of G and is denoted by $d_\chi(G)$. In this paper, we prove that if G is a claw-free graph with minimum degree at least two, then $d_\chi(G) \geq 2$.

Keywords: Chromatic number, dominating set, maximal independent set, dominating-χ-color number, claw-free graph

1 Introduction

Throughout this paper all graphs are simple. Let G be a graph with the vertex set $V(G)$ and the edge set $E(G)$. For any vertex $v \in V(G)$, the *open neighborhood* v, denoted by $N(v)$, is the set $\{u \in V(G) \mid uv \in E(G)\}$ and the *closed neighborhood*, denoted by $N[v]$ is $N(v) \cup \{v\}$. We denote the degree of v by $d(v)$ which is $|N(v)|$ and the *minimum degree* of G is denoted by $\delta(G)$. The vertex v is called *pendant* if $d(v) = 1$. For a set $S \subseteq V$, its open neighborhood is $N(S) = \bigcup_{v \in S} N(v)$ and its closed neighborhood is $N[S] = N(S) \cup S$. In the graph G, a set $S \subseteq V(G)$ is a *dominating set* if every vertex not in S has a neighbor in S. Let G be a graph and $X \subseteq V(G)$. We define $G[X]$ as the induced subgraph on X. The maximum cardinality of $X \subseteq V(G)$ where $G[X]$ has no edges is called the *the independent number* of G and denoted by $\alpha(G)$. A graph is called *claw-free* if it has no claw as an induced subgraph, where *claw* is a graph that isomorphic to $K_{1,3}$.

The *corona* of a graph H, $cor(H)$, is that the graph obtained from H by adding a pendant edge to each vertex of H. Cycle and complete graph of order n are denoted by C_n and K_n, respectively. Also we use $P = (v_1, \ldots, v_n)$ as a path with the vertex set $\{v_1, \ldots, v_n\}$ and the edge set $E(G) = \{v_i v_{i+1} \mid 1 \leq i \leq n-1\}$.

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A *proper vertex coloring* of a graph is an assignment of colors to the vertices so that adjacent vertices get different colors. Since all coloring in this paper are proper coloring, we simply call a proper coloring a coloring. The *chromatic number* $\chi(G)$ is the minimum number of colors needed to color graph properly. In a given coloring of the graph, a set consisting of all those vertices assigned the same color is called a *color class*. These color classes partition V into independent sets, not necessarily dominating sets. If $\mathcal{C}$ is a coloring of the graph with color classes $U_1, \ldots, U_{\chi(G)}$, then we write $\mathcal{C} = (U_1, \ldots, U_{\chi(G)})$.

Among all χ-colorings of G, let $\mathcal{C}$ be chosen to have a color class U_1 that dominates as many vertices of G as possible. If there is a vertex in G not dominated by U_1, then deleting such a vertex from its class in $\mathcal{C}$ and adding it to the color class U_1 produces a new coloring that contains a color class relating to color 1 which dominates more vertices than U_1, a contradiction. Hence the color class U_1 is a dominating set in G. Therefore every graph G contains a coloring with the property that the first class is a dominating set in G. This first observed in [4] motivated Arumugan et al. [2] to define the dominating-χ-color number. Among all coloring of G, a coloring with the maximum number of color classes that are dominating sets in G is defined to be the *dominating-χ-coloring* of G. The number of color classes that are dominating sets in a dominating-χ-coloring of G is defined to be the *dominating-χ-color number* of G, denoted by $d_{\chi(G)}$. According to the mentioned observation, among dominating-χ-coloring of G, there is a coloring $\mathcal{C} = (U_1, \ldots, U_{\chi(G)})$ so that U_1 and U_2 dominate $V(G)$ and $V(G)\backslash U_1$, respectively. Therefore we call a χ-coloring $\mathcal{C} = (U_1, \ldots, U_{\chi(G)})$ as *hierarchical coloring of order t* if and only if U_1 is a dominating set and for each i, $2 \leq i \leq t$, the set U_i dominates $V(G)\backslash \bigcup_{j=1}^{i-1} U_j$. Obviously, for every t, $1 \leq t \leq \chi(G)$, the set of $\chi(G)$-colorings of G contains at least one hierarchical coloring of order t.

There has been a great deal of interest in relating graph coloring and the dominating set which have been well studied; for example see [3]. The dominating-χ-color number was first aforementioned in [2]. Arumugam et al. [1] presented conditions on a graph G satisfying $d_\chi(G) = 1$, moreover, they provided an upper bound on the dominating-χ-color number. They proved that for every pair of integers (k, l) there exists a connected graph G with $\chi(G) = k$ and $d_\chi(G) = l$. In this paper, we show that if G is a claw-free graph and $\delta(G) \geq 2$, then the dominating-χ-color number of G is at least 2.

The following results were proved in [2].

Theorem 1.1 *For all graph G, $1 \leq d_\chi(G) \leq \delta(G) + 1$.*

Theorem 1.2 *Let G be a graph of order n with no isolated vertex. Then,* *(i)* $\chi(G) = n$ *if and only if* $d_\chi(G) = n$ *if and only if* $G = K_n$.

(ii) If G *is bipartite, then* $d_\chi(G) = 2$.

Theorem 1.3 [1] *For* $n \geq 3$,

$$d_\chi(C_n) = \begin{cases} 3 \text{ ifn} \equiv 3 \pmod 6, \\ 2 \quad \textit{otherwise.} \end{cases}$$

Let $\mathcal{C} = (U_1, \ldots, U_{\chi(G)})$ be a vertex coloring of G and $u \in U_i, v \in U_j$, for some $1 \leq i, j \leq \chi(G)$. Let $MP_{i,j}(u, v)$ be the set of all maximal paths starting at u, v is the second vertex of these paths and other vertices of these paths belong to the color classes U_i and U_j, alternatively.

2 Results

The main goal of this paper is showing that the dominating-χ-color number of every claw-free graph with no pendant edge is at least 2. Before proving the main result, we prove the following lemmas.

Lemma 2.1 *Let* G *be a claw-free graph with hierarchical coloring* $\mathcal{C} = (U_1, \ldots, U_{\chi(G)})$ *of order* 2. *Let* $a \in U_1$ *and* $N(a) \cap U_2 = \varnothing$. *If there exists* $u \in N(a)$ *such that* $N(u) \cap U_1 = \{a\}$, *then* G *contains a hierarchical coloring* $\widetilde{\mathcal{C}} = (\widetilde{U}_1, \ldots, \widetilde{U}_{\chi(G)})$ *of order* 2 *such that* $|N[\widetilde{U}_2]| > |N[U_2]|$.

Lemma 2.2 *Let* G *be a claw-free graph with a hierarchical coloring* $\mathcal{C} = (U_1, \ldots, U_{\chi(G)})$ *of order* 2 *and* $a \in U_1$. *If the coloring satisfies two following conditions:*

(i) $N(a) \cap U_2 = \varnothing$ *and* $|N(u) \cap U_1| \geq 2$, *for every* $u \in N(a)$,

(ii) There exists a path $P \in \bigcup_{u \in N(a)} MP_{1,c(u)}(a, u)$ *with one endpoint not in* U_1,

then G *contains a hierarchical coloring* $\widetilde{\mathcal{C}} = (\widetilde{U}_1, \ldots, \widetilde{U}_{\chi(G)})$ *of order* 2 *such that* $|N[\widetilde{U}_2]| > |N[U_2]|$.

Lemma 2.3 *Let* G *be a claw-free graph with a hierarchial coloring* $\mathcal{C} = (U_1, \ldots, U_{\chi(G)})$ *of order* 2, $a \in U_1$ *such that* $N(a) \cap U_2 = \varnothing$ *and* $|N(u) \cap U_1| \geq 2$, *for every* $u \in N(a)$. *Assume that two endpoints of every path* $P \in \bigcup_{u \in N(a)} MP_{1,c(u)}(a, u)$ *belong to* U_1. *If* $P = (a, v_1, a_1, \ldots, v_r, a_r)$ *and*

$$\widetilde{U}_1 = (U_1 \backslash (\{a\} \cup \{a_i | 1 \leq i \leq r\})) \cup \{v_i | 1 \leq i \leq r\} \cup J,$$
$$\widetilde{U}_2 = U_2 \cup \{a\} \cup \{a_i \mid N(a_i) \cap U_2 = \varnothing, 1 \leq i \leq r\},$$

$\widetilde{U}_{c(v_1)} = (U_{c(v_1)} \backslash \{v_i \mid 1 \leq i \leq r\}) \cup \{a_i \mid N(a_i) \cap U_2 \neq \varnothing, 1 \leq i \leq r\}$,

$\widetilde{U}_i = U_i \backslash J$, *for* $i \in \{3, 4, \ldots, \chi(G)\} \backslash \{c(v_1)\}$,

where J *is a maximal independent set of* $\mathcal{J} = \{z \in V(G) \mid \{a, a_r\} \subseteq N(z), N[z] \cap \{v_1, \ldots, v_r\} = \varnothing\}$, *then* $\widetilde{\mathcal{C}} = (\widetilde{U}_1, \ldots, \widetilde{U}_{\chi(G)})$ *is a coloring such that*

(i) $\widetilde{U}_1$ *dominates* $V(G) \backslash \{v \in N(a_r) \mid |N(v) \cap U_1| = 1$ *and* $N(v) \cap (\{v_i | 1 \leq i \leq r\} \cup J) = \varnothing\}$.

(ii) $\widetilde{U}_2$ *dominates* $V(G) \backslash \widetilde{U}_1$.

Lemma 2.4 *Let* G *be a claw-free graph with a hierarchical coloring* $\mathcal{C} = (U_1, \ldots, U_{\chi(G)})$ *of order* 2 *and* $a \in U_1$. *If the coloring satisfies three following conditions,*

(i) $N(a) \cap U_2 = \varnothing$ *and* $|N(u) \cap U_1| \geq 2$, *for every* $u \in N(a)$,

(ii) Two endpoints of each path in $\bigcup_{u \in N(a)} MP_{1,c(u)}(a, u)$ *belong to* U_1,

(iii) There exists $P = (a, v_1, a_1, \ldots, v_r, a_r) \in \bigcup_{u \in N(a)} MP_{1,c(u)}(a, u)$ *with endpoints* a *and* a_r *such that at least one of the following holds,*

(1) $N(a_r) \cap U_2 = \varnothing$,

(2) If $h \in N(a_r) \cap U_2$, *then* $|N(h) \cap U_1| \geq 2$,

then G *contains a hierarchical coloring* $\widetilde{\mathcal{C}} = (\widetilde{U}_1, \ldots, \widetilde{U}_{\chi(G)})$ *of order* 2 *with* $|N[\widetilde{U}_2]| > |N[U_2]|$.

Lemma 2.5 *Let* G *be a claw-free graph with a hierarchical coloring* $\mathcal{C} = (U_1, \ldots, U_{\chi(G)})$ *of order* 2, $\delta(G) \geq 2$ *and* $a \in U_1$. *If* $\mathcal{C}$ *satisfies three following conditions:*

(i) $N(a) \cap U_2 = \varnothing$ *and* $|N(u) \cap U_1| \geq 2$, *for every* $u \in N(a)$,

(ii) Two endpoints of each path in $\bigcup_{u \in N(a)} MP_{1,c(u)}(a, u)$ *belong to* U_1,

(iii) For each $P = (a, v_1, a_1, \ldots, v_r, a_r) \in \bigcup_{u \in N(a)} MP_{1,c(u)}(a, u)$ *with endpoint* a_r, *there is a vertex* $h \in N(a_r) \cap U_2$ *such that* $|N(h) \cap U_1| = 1$,

then G *contains a hierarchical coloring* $\widetilde{\mathcal{C}} = (\widetilde{U}_1, \ldots, \widetilde{U}_{\chi(G)})$ *of order* 2 *such that* $|N[\widetilde{U}_2]| > |N[U_2]|$.

Now, we are in a position to prove the main theorem.

Theorem 2.6 *If* G *is a claw-free graph and* $\delta(G) \geq 2$, *then* $d_\chi(G) \geq 2$.

Proof. Among all hierarchical coloring of order 2, let $\mathcal{C} = (U_1, \ldots, U_{\chi(G)})$ be a coloring in which $|N[U_2]|$ is maximum. We claim that U_2 dominates $V(G)$.

Assume, to the contrary that there is a vertex $a \in V(G)$ which is not dominated by U_2. Hence $a \in U_1$ and $N(a) \cap U_2 = \varnothing$. Since $d(a) \geq 2$, $N(a) \neq \varnothing$. Now, we have four possibilities:

Case (i) There is $u \in N(a)$ such that $N(u) \cap U_1 = \{a\}$. By Lemma 2.1, G contains a hierarchical coloring $\widetilde{\mathcal{C}} = (\widetilde{U}_1, \ldots, \widetilde{U}_{\chi(G)})$ of order 2 with $|N[\widetilde{U}_2]| > |N[U_2]|$. This is a contradiction.

Case (ii) For every $u \in N(a)$, $|N(u) \cap U_1| \geq 2$ and there is a path $P \in \bigcup_{u \in N(a)} MP_{1,c(u)}(a, u)$ whose one of the endpoints is not in U_1. By Lemma 2.2, G contains a hierarchical coloring $\widetilde{\mathcal{C}} = (\widetilde{U}_1, \widetilde{U}_2, \ldots, \widetilde{U}_t)$ of order 2 with $|N[\widetilde{U}_2]| > |N[U_2]|$. This is a contradiction.

Case (iii) For every $u \in N(a)$, $|N(u) \cap U_1| \geq 2$ and two endpoints of each path in $\bigcup_{u \in N(a)} MP_{1,c(u)}(a, u)$ belong to U_1 and there is $P \in \bigcup_{u \in N(a)} MP_{1,c(u)}(a, u)$ with endpoints a and a_r such that $N(a_r) \cap U_2 = \varnothing$. By Lemma 2.4, G contains a hierarchical coloring $\widetilde{\mathcal{C}} = (\widetilde{U}_1, \widetilde{U}_2, \ldots, \widetilde{U}_{\chi(G)})$ of order 2 such that $|N[\widetilde{U}_2]| > |N[U_2]|$. This is a contradiction.

Case (iv) For every $u \in N(a)$, $|N(u) \cap U_1| \geq 2$ and two endpoints of each path in $\bigcup_{u \in N(a)} MP_{1,c(u)}(a, u)$ belong to U_1 and there is $P \in \bigcup_{u \in N(a)} MP_{1,c(u)}(a, u)$ with endpoints a and a_r such that if $u \in N(a_r) \cap U_2$, then $|N(u) \cap U_1| \geq 2$. By Lemma 2.4, G contains a hierarchical coloring $\widetilde{\mathcal{C}} = (\widetilde{U}_1, \widetilde{U}_2, \ldots, \widetilde{U}_{\chi(G)})$ of order 2 such that $|N[\widetilde{U}_2]| > |N[U_2]|$. This is a contradiction.

Case (v) For every $u \in N(a)$, $|N(u) \cap U_1| \geq 2$, two endpoints of each path in $\bigcup_{u \in N(a)} MP_{1,C(u)}(a, u)$ belong to U_1 and for each $P \in \bigcup_{u \in N(a)} MP_{1,C(u)}(a, u)$ with endpoints a and a_r, if $u \in N(a_r) \cap U_2$ then $N(u) \cap U_1 = \{a_r\}$. By Lemma 2.5, G contains a hierarchical coloring $\widetilde{\mathcal{C}} = (\widetilde{U}_1, \widetilde{U}_2, \ldots, \widetilde{U}_{\chi(G)})$ from order 2 with $|N[\widetilde{U}_2]| > |N[U_2]|$. This is a contradiction. □

We close the paper with two following remarks.

Remark 2.7 *The lower bound in Theorem 2.6 is tight. For instance, consider* C_n *with* $n \not\equiv 3 \pmod{6}$. *Theorem 1.3 shows that* $d_{\chi(G)} = 2$.

Remark 2.8 *The condition* $\delta(G) \geq 2$ *in Theorem 2.6 is necessary. If* $G = cor(K_n)$, *then* $\delta(cor(K_n)) = 1$ *and* $d_\chi(G) = 1$.

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