Reflective Guarding a Gallery

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Abstract. This paper studies a variant of the Art Gallery problem in which the "walls" can be replaced by *reflecting edges*, which allows the guards to see further and thereby see a larger portion of the gallery. Given a simple polygon P, first, we consider one guard as a point viewer, and we intend to use reflection to add a certain amount of area to the visibility polygon of the guard. We study visibility with specular and diffuse reflections where the specular type of reflection is the mirror-like reflection, and in the diffuse type of reflection, the angle between the incident and reflected ray may assume all possible values between 0 and π. Lee and Aggarwal already proved that several versions of the general Art Gallery problem are *NP*-hard. We show that several cases of adding an area to the visible area of a given point guard are *NP*-hard, too.
Second[1], we assume that all edges are reflectors, and we intend to decrease the minimum number of guards required to cover the whole gallery. Chao Xu proved that even considering r specular reflections, one may need $\lfloor \frac{n}{3} \rfloor$ guards to cover the polygon. Let r be the maximum number of reflections of a guard's visibility ray.
In this work, we prove that considering r *diffuse* reflections, the minimum number of *vertex or boundary* guards required to cover a given simple polygon $\mathcal{P}$ decreases to $\lceil \frac{\alpha}{\mathbf{1}+\lfloor \frac{\mathbf{r}}{\mathbf{8}} \rfloor} \rceil$, where α indicates the minimum number of guards required to cover the polygon without reflection. We also generalize the $\mathcal{O}(\log n)$-approximation ratio algorithm of the vertex guarding problem to work in the presence of reflection.

1 Introduction

Consider a simple polygon $\mathcal{P}$ with n vertices and a point viewer q inside $\mathcal{P}$. Suppose $C(\mathcal{P})$ denotes $\mathcal{P}$'s topological closure (the union of the interior and the boundary of $\mathcal{P}$). Two points x and y are visible to each other, if and only if the

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[1] A primary version of the second result presented here is accepted in EuroCG 2022 [1] whose proceeding is not formal.

line segment $\overline{xy}$ lies completely in $C(\mathcal{P})$. The visibility polygon of q, denoted as $VP(q)$, consists of all points of $\mathcal{P}$ visible to q. Many problems concerning visibility polygons have been studied so far. There are linear-time algorithms to compute $VP(q)$ ([2], [3]). Edges of $VP(q)$ that are not edges of $\mathcal{P}$ are called *windows*.

If some of the edges of $\mathcal{P}$ are made into mirrors, then $VP(q)$ may enlarge. Klee first introduced visibility in the presence of mirrors in 1969 [4]. He asked whether every polygon whose edges are all mirrors is illuminable from every interior point. In 1995 Tokarsky constructed an all-mirror polygon inside which there exists a dark point [5]. Visibility with reflecting edges subject to different types of reflections has been studied earlier [6]: (1) *Specular-reflection*: in which the direction light is reflected is defined by the law-of-reflection. Since we are working in the plane, this law states that the angle of incidence and the angle of reflection of the visibility rays with the normal through the polygonal edge are the same. (2) *Diffuse-reflection*: that is to reflect light with all possible angles from a given surface. The diffuse case is where the angle between the incident and reflected ray may assume all possible values between 0 and π.

Some papers have specified the maximum number of allowed reflections via mirrors in between [7]. In multiple reflections, we restrict the path of a ray coming from the viewer to turn at polygon boundaries at most r times. Each time this ray will reflect based on the type of reflection specified in a problem (specular or diffuse).

Every edge of $\mathcal{P}$ can potentially become a reflector. However, the viewer may only see some edges of $\mathcal{P}$. When we talk about an edge, and we want to consider it as a reflector, we call it a *reflecting edge* (or a *mirror-edge* considering specular reflections). Each edge has the potential of getting converted into a reflecting edge in a final solution of a visibility extension problem (we use the words "reflecting edge" and "reflected" in general, but the word "mirror" is used only when we deal with specular reflections).

Two points x and y inside $\mathcal{P}$ can see each other through a reflecting edge e, if and only if they are reflected visible with a specified type of reflection. We call these points *reflected visible* (or *mirror-visible*).

The Art Gallery problem is to determine the minimum number of guards that are sufficient to see every point in the interior of an art gallery room. The art gallery can be viewed as a polygon $\mathcal{P}$ of n vertices, and the guards are stationary points in $\mathcal{P}$. If guards are placed at vertices of $\mathcal{P}$, they are called *vertex guards*. If guards are placed at any point of $\mathcal{P}$, they are called *point guards*. If guards are allowed to be placed along the boundary of $\mathcal{P}$, they are called *boundary-guards* (on the perimeter). To know more details on the history of this problem see [8].

The Art Gallery problem was proved to be *NP*-hard first for polygons with holes by [9]. For guarding simple polygons, it was proved to be *NP*-complete for vertex guards by [10]. This proof was generalized to work for point guards by [11]. The class $\exists\mathbb{R}$ consists of problems that can be reduced in polynomial time to the problem of deciding whether a system of polynomial equations with integer coefficients and any number of real variables has a solution. It can be easily seen that $NP \subseteq \exists\mathbb{R}$. The article [12] proved that the Art Gallery problem is $\exists\mathbb{R}$-complete. Sometimes irrational coordinates are required to describe an optimal solution [13].

Ghosh [14] provided an $\mathcal{O}(\log n)$-approximation algorithm for guarding polygons with or without holes with *vertex* guards. King and Kirkpatrick obtained an approximation factor of $\mathcal{O}(\log\log(OPT))$ for vertex guarding or perimeter guarding simple polygons [15]. To see more information on approximating various versions of the Art Gallery problem see [16], or [17].

Result 1 *Given a simple polygon $\mathcal{P}$ and a query point q as the position of a single viewer (guard), consider extending the area of the visibility polygon of q ($VP(q)$) by choosing an appropriate subset of edges and make them reflecting edges so that q can see the whole $\mathcal{P}$.*

A) To extend the surface area of $VP(q)$ by exactly a given amount, the problem is NP-complete.

B) To extend the surface area of $VP(q)$ using the minimum number of diffuse reflecting edges and by at least a given amount, the problem is NP-hard.

Result 2 *Suppose that in the Art Gallery problem a given polygon, possibly with holes, can be guarded by α vertex guards without reflections, then the gallery can be guarded by at most $\lceil \frac{\alpha}{1+\lfloor \frac{r}{8} \rfloor} \rceil$ guards when r diffuse reflections are permitted.*

For both the diffuse and specular reflection the Art Gallery problem considering r diffuse reflection is solvable in $\mathcal{O}(n^{8^{r+1}+10})$ time with an approximation ratio of $\mathcal{O}(\log n)$.

1.1 Our Settings

Every guard can see a point if the point is directly visible to the guard or if it is reflected visible. This is a natural and non-trivial extension of the classical Art Gallery setting. The problem of visibility via reflection has many applications in wireless networks, and Computer Graphics, in which the signal and the view ray can reflect on walls several times, and it loses its energy after each reflection. There is a large literature on geometric optics (such as [18], [19], [20]), and on the chaotic behavior of a reflecting ray of light or a bouncing billiard ball (see, e.g., [21], [22], [23], [24]). Particularly, regarding the Art Gallery problem, reflection helps in decreasing the number of guards (see Figure 1).

Sections 2 and 3 study the problem of extending the surface area of the visibility polygon $VP(q)$ of a point guard q inside a polygon $\mathcal{P}$ by means of reflecting edges. Section 3 considers the scenario in which the visibility polygon of the source needs to be extended at least k units of area where k is a given value. However, to make the problem more straightforward, one may consider adding a specific area with an exact given surface area to the visibility of the source. Section 2 considers extending the visibility polygon of q exactly k units of area.

A special reflective case of the general Art Gallery problem is described by Chao Xu in 2011 [25]. Since we want to generalize the notion of guarding a

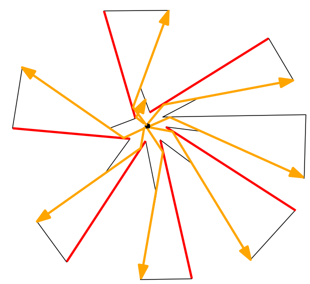

Fig. 1. This figure illustrates a situation where a single guard is required if we use reflection-edges; $\Theta(n)$ guards are required if we do not consider reflection. Red segments illustrate the reflected-edges.

simple polygon, if the edges become mirrors instead of walls, the light loses intensity every time it gets reflected on the mirror. Therefore after r reflections, it becomes undetectable to a guard. Chao Xu proved that regarding multiple specular reflections, for any n, there exist polygons with n vertices that need $\lfloor \frac{n}{3} \rfloor$ guards. Section 4 deals with the same problem but regarding diffuse reflection. G. Barequet et al. [26] proved the minimum number of diffuse reflections sufficient to illuminate the interior of any simple polygon with n walls from any interior-point light source is $\lfloor \frac{n}{2} \rfloor - 1$. E. Fox-Epstein et al. [27] proved that to make a simple polygon in a general position visible for a single point light source, we need at most $\lfloor \frac{n-2}{4} \rfloor$ diffuse reflections on the edges of the polygon, and this is the best possible bound. These two papers consider a *single point viewer*; however, in Section 4 we considered helping the art gallery problem with diffuse reflection on the edges of the polygon. So, we want to decrease the minimum number of guards required to cover a given simple polygon using r diffuse reflections. We will prove that we can reduce the optimal number with the help of diffuse reflection. Note that we do not assume general positions for the given polygon. For more information on combining reflection with the art gallery problem see [28], [29], [30], [7], and [6].

2 Expanding $VP(q)$ by *exactly* k units of area

We begin this section with the following theorem, and the rest of the section covers the proof of this theorem.

Theorem 1. *Given a simple polygon $\mathcal{P}$, a point $q \in \mathcal{P}$, and an integer $k > 0$, the problem of choosing any number of, say l, reflecting edges of $\mathcal{P}$ in order to expand $VP(q)$ by* exactly *k units of area is NP-complete in the following cases:*

1. *Specular-reflection where a ray can be reflected only once.*
2. *Diffuse-reflection where a ray can be reflected any number of times.*

Clearly, it can be verified in polynomial time if a given solution adds precisely k units to $VP(q)$. Therefore, the problem is in NP.

Consider an instance of the Subset-Sum problem (*InSS*), which has $val(1), val(2)$ $..., val(m)$ non-negative integer values, and a target number $\mathcal{T}$. Suppose $m \in \Theta(n)$, where n indicates the number of vertices of $\mathcal{P}$. The Subset-Sum Problem involves determining whether a subset from a list of integers can sum to a target value $\mathcal{T}$. Note that the variant in which all inputs are positive is NP-complete as well [31].

In the following subsections, we will show that the Subset-Sum problem is reducible to this problem in polynomial time. Thus, we deduce that our problem in the cases mentioned above is *NP*-complete.

2.1 *NP*-hardness for specular reflections

The reduction polygon $\mathcal{P}$ consists of two rectangular chambers attached side by side. The chamber to the right is taller, while the chamber to the left is shorter but quite broad. The query point q is located in the right chamber (see Figure 2). The left chamber has left-leaning triangles attached to its top and bottom edges. In the reduction from Subset-Sum, the areas of the bottom spikes correspond to the weights of the sets (the values of *InSS*). The top triangles are narrow and have negligible areas. Their main purpose is to house the edges which may be turned into reflecting edges so that q can see the bottom spikes.

To describe the construction formally, consider *InSS*. Denote the i^{th} value by $val(i)$ and the sum of the values till the i^{th} value, $\sum_{k=1}^{i} val(k)$, by $sum(i)$. We construct the reduction polygon in the following steps:

(1) Place the query point q at the origin $(0, 0)$.

(2) Consider the x-axis as the bottom edge of the left rectangle.

(3) Denote the left, right and bottom points of the i^{th} bottom spike by $llt(i)$, $rlt(i)$ and $blt(i)$ respectively. Set the coordinates for $llt(i)$ at $(i+2(sum(i-1)), 0)$, $rlt(i)$ at $(i + 2(sum(i)), 0)$, and those for $blt(i)$ at $(i + 2(sum(i)), -1)$.

(4) The horizontal polygonal edges between the top triangles are good choices for mirrors, so we call them *mirror-edges*. The i^{th} mirror-edge lies between the $(i-1)^{th}$ and i^{th} top spikes. Denote the left and right endpoints of the i^{th} mirror-edge by $lm(i)$ and $rm(i)$ respectively. Set the coordinates of $lm(i)$ at $(\frac{(i+2(sum(i-1)))}{2}, 2(sum(m)+m))$, and those of $rm(i)$ at $(\frac{(i+2(sum(i)))}{2}, 2(sum(m)+m))$.

Denote the topmost point of the i^{th} top spike by $ut(i)$ and set its coordinates at $(\frac{(i+2(sum(i)))}{2}, 4(sum(m) + 2m))$.

2.2 Properties of the reduction polygon

In this subsection, we discuss properties that follow from the above construction of the reduction polygon. We have the following lemmas.

Lemma 1. *The query point q can see the region enclosed by the i^{th} bottom spike only through a specular reflection through the i^{th} mirror-edge.*

Proof. See the full version of the paper [32] for the proof.

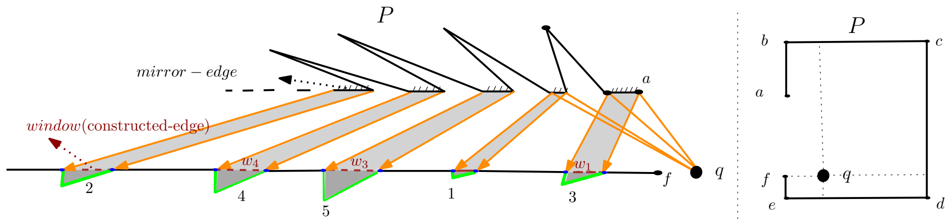

Fig. 2. Two main components of the reduction polygon is illustrated.

Lemma 2. *All coordinates of the reduction polygon are rational and take polynomial time to compute.*

Proof. This too follows from the construction, as the number of sums used is linear, and each coordinate is derived by at most one division from such a sum.

Lemma 3. *The problem of extending the visibility polygon of a query point inside a simple polygon via single specular reflection is NP-complete.*

Proof. From Lemma 1 it follows that a solution for the problem exists if an only if a solution exists for the corresponding Subset-Sum problem. From Lemma 2 it follows that the reduction can be carried out in polynomial time, thus proving the claim.

Observation 1 *The multiple reflection case of the first case of the problem mentioned in Theorem 1 is still open. That is the above-mentioned reduction (presented in subsection 2.1) does not work if more than one reflection is allowed.*

Proof. See the proof in the complete version of the paper [32].

2.3 *NP*-hardness for diffuse reflections

This subsection deals with the second part of Theorem 1. Considering diffuse reflection, since rays can be reflected into wrong spikes (a spike which should get reflected visible via another reflecting edge) the previous reduction does not work. Considering multiple plausible reflections, the problem becomes even harder. These rays have to be excluded by an appropriate structure of the polygonal boundary.

The construction presented in this subsection works in the case of multiple reflections, too. Again we reduce the Subset-Sum problem to our problem (Result 1(A) considering diffuse reflections).

As before, we place the query point q at the origin, $(0, 0)$. The main polygon $\mathcal{P}$ used for the reduction is primarily a big rectangle, with around two-thirds of it being to the right of q (see Figure 3). On top of this rectangle are m *"double triangle"* structures. Each double triangle structure consists of triangles sharing some of their interiors. The lower among these two triangles, referred to as the *second triangles*, is right-angled and has its base on the main rectangle, with its

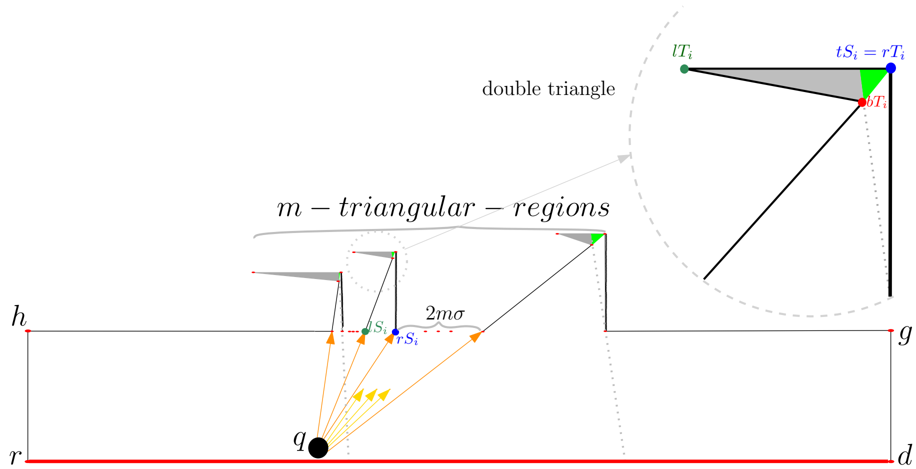

Fig. 3. A schema of the reduction polygon. Note that $m \in \Theta(n)$. The main polygon is a rectangle, with gadgets on its top edge. The green region plus the grey region are in one triangle which its surface area equals to a value of *InSS*.

altitude to the right and hypotenuse to the left. The upper triangle, referred to as the *"top triangle"*, is inverted, i.e., its base or horizontal edge is at the top. One of its vertices is the top vertex of the second triangle, and another of its vertices merges into the hypotenuse of the second triangle. Its third vertex juts out far to the left, at the same vertical level as the top vertex of the second triangle, making the top triangle a very narrow triangle. The area of the top triangle equals the value of the i^{th} set in the Subset-Sum problem (val_i).

We make each top triangle to get diffusely reflected visible by only some specific reflecting edges. However, as mentioned previously, there can be a troublemaker shared reflected visible area in each top triangle. This area is illustrated in green in Figure 3. We know that every value of the Subset-Sum problem is an integer. We manage to set the coordination of the polygon so that the sum of the surface area of all the green regions gets equal to a value less than 1, and all of these regions are entirely reflected visible to the lower edge of the main rectangle. As a result, seeing the green areas through a reflection via the bottom red edge cannot contribute towards seeing exactly an extra region of k units of area. Remember that k is an integer.

Formally, denote the ith top and second triangles by T_i and S_i respectively. Denote the top, left and right vertices of S_i by tS_i, lS_i and rS_i respectively. Denote the sum of values of all subsets of the Subset-Sum problem by σ. In fact, $\sigma = \sum_{i=1}^{m} val_i$. In general, the triangle S_i has a base length of i units, and its base is $2m^2\sigma$ units distance away from those of S_{i-1} and S_{i+1}. Therefore the coordinates of lS_i and rS_i are given by $((2m^2\sigma)\frac{i(i+1)}{2} - i, m^2(m+1)\sigma)$ and $((2m^2\sigma)\frac{i(i+1)}{2}, m^2(m+1)\sigma)$ respectively. For any vertex v of the reduction polygon, let us denote the x and y coordinates of v by $x(v)$ and $y(v)$ respectively.

The vertex tS_i is obtained by drawing a ray originating at q and passing through lS_i, and having it intersect with the vertical line passing through rS_i. This point of intersection is tS_i with coordinates $(x(rS_i), m^2(m+1)\sigma + \frac{m^2(m+1)\sigma}{x(lS_i)})$.

Denote the leftmost and bottom-most vertices of T_i by lT_i and bT_i respectively. Recall that bT_i lies on the hypotenuse of S_i, and lT_i has the same y-coordinate as tS_i. Moreover, we place bT_i in such a way, that the sum of the total regions of all top triangles seen from the base of the main rectangle (the rd edge in Figure 3) is less than 1. Intuitively, bT_i divides the hypotenuse of S_i in the $m^2-1:1$ ratio. Accordingly, the coordinates of bT_i are: $(1+\frac{i(i-1)}{2}+(m^2-1)\frac{(x(tS_i)-x(lS_i))}{(m^2)}, m+(m^2-1)\frac{(y(tS_i)-y(lS_i))}{(m^2)})$.

Next, we set coordinates of lT_i in a way that the total surface area of T_i gets equal to the value of the i^{th} subset. Denote the value of the i^{th} subset by val_i. Then, the coordinates of lT_i are given by $(x(tS_i) - 2\frac{val_i}{y(tS_i)-y(bT_i)}, y(tS_i))$. Finally, the coordinates of the four vertices of the main rectangle holding all the double triangle gadgets, are given by $(-x(rS_m), m^2(m+1)\sigma)$, $(-x(rS_m), -1)$, $(2(x(rS_m)), -1)$ and $(2(x(rS_m)), m^2(m+1)\sigma)$.

Lemma 4. *The reduction stated in subsection 2.3 proves that the problem of adding exactly k units of area to a visibility polygon via only a single diffuse-reflection per ray, is NP-complete. The reduction polygon has rational coordinates with size polynomial with respect to n.*

Proof. See the full version of the paper [32] for the proof.

We can use the above-mentioned reduction in case multiple reflection is allowed. See the following corollary.

Corollary 2 *The reduction stated in subsection 2.3 works if multiple reflections is allowed.*

Proof. See the full version of the paper [32] for the proof.

3 Expanding at least k units of area

In this section, we modify the reduction of Lee and Lin [10] and use it to infer that the problem of extending the visibility polygon of a given point by a region of area at least k units of area with the *minimum* number of reflecting edges is *NP*-hard, where k is a given amount (Result 1(B)). The idea is that the potential vertex guards are replaced with edges that can reflect the viewer (q). We need an extra reflecting edge, though. Only a specific number of edges can make an invisible region (a spike) entirely reflected visible to the viewer. Converting the correct minimum subset of these edges to the reflecting edges determines the optimal solution for the problem.

The specular-reflection cases of the problem are still open. Nonetheless, it was shown by Aronov [6] in 1998 that in such a polygon where all of its edges are mirrors, the visibility polygon of a point can contain holes. And also, when we consider at most r specular reflections for every ray, we can compute the visibility polygon of a point inside that within $\mathcal{O}(n^{2r} \log n)$ of time complexity and $\mathcal{O}(n^{2r})$ of space complexity [7].

Conjecture 1. Given a simple polygon $\mathcal{P}$, and a query point q inside the polygon, and a positive value k, the problem whether l of the edges of the polygon can be turned to reflecting edges so that the area added to $VP(q)$ through (single/multiple) diffuse-reflections increases at least k units of area is NP-hard.

Proof Idea. See the full version of the paper for the proof [32].

4 Regular Visibility vs Reflection

This section deals with Result 2. Under some settings, visibility with reflections can be seen as a general case of regular visibility. For example, consider guarding a polygon $\mathcal{P}$ with vertex guards, where all the edges of the polygons are diffuse reflecting edges, and r reflections are allowed for each ray. Let S be the set of guards in an optimal solution if we do not consider reflection. Since we allow multiple vertices of the polygon to be collinear, we slightly change the notion of a visibility polygon for convenience. Consider the visibility polygon of a vertex (see Figure 4). It may include lines containing points that do not have any interior point of the visibility polygon within a given radius. So, given a vertex x of $\mathcal{P}$, we consider the union of the interior of the original $VP(x)$ (denoted by $int\,VP(x)$), and the limit points of $int\,VP(x)$, as our new kind of visibility polygon of x. Clearly, a guard set of $\mathcal{P}$ gives a set of the new kind of visibility polygons whose union is $\mathcal{P}$.

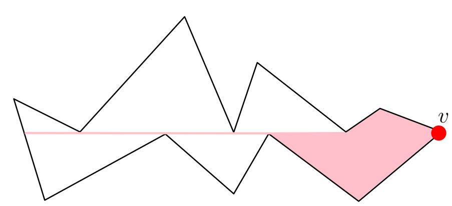

Fig. 4. The visibility polygon of a point v.

Consider any guard $v \in S$. The visibility polygon $VP(v)$ of v must have at least one window[2]. Otherwise, v is the only guard of $\mathcal{P}$. Consider such a window, say, w. Let x be a point of intersection of w with the polygonal boundary. Then there must be at least another guard $u \in S$ such that x lies in both $VP(u)$ and $VP(v)$. The following lemmas discuss how $VP(u)$ and $VP(v)$ can be united using a few diffuse reflections, and how the whole polygon can be seen by a just a fraction of the optimal guards, depending on the number of reflections allowed per ray.

[2] A window is an edge of a viewer's visibility polygon, which is not a part of an edge of the main polygon.

Theorem 3. *If $\mathcal{P}$ can be guarded by α vertex guards without reflections, then $\mathcal{P}$ can be guarded by at most $\lceil \frac{\alpha}{1+\lfloor \frac{r}{8} \rfloor} \rceil$ guards when r diffuse reflections are permitted.*

To prove this theorem see the following lemmas first:

Lemma 5. *If $\mathcal{S}$ is an optimal vertex guard set of polygon $\mathcal{P}$ and $|\mathcal{S}| > 1$ then for every guard $u \in \mathcal{S}$ there exists a different guard $v \in \mathcal{S}$" such that u and v can see each other through five diffuse reflections. Furthermore, u and v can fully see each other's visibility polygons with eight diffuse reflections.*

Proof. See the full version of the paper [32] for the proof.

Now we build a graph G as follows. We consider the vertex guards in $\mathcal{S}$ as the vertices of G, and add an edge between two vertices of G if and only if the two corresponding vertex guards in $\mathcal{S}$ can see each other directly or through at most five reflections. We have the following Lemma.

Lemma 6. *The graph G is connected.*

Proof. See the full version of the paper [32] for the proof.

Consider any optimal vertex guard set $\mathcal{S}$ for the Art Gallery problem on the polygon $\mathcal{P}$, where $| \mathcal{S} |= \alpha$. Build a graph G on $\mathcal{S}$ as it was mentioned before Lemma 6. Due to Lemma 6, G is connected. Find a spanning tree T of G and root it at any vertex. Denote the i^{th} level of vertices of T by L_i. Given a value of r, divide the levels of T into $1 + \lfloor \frac{r}{8} \rfloor$ classes, such that the class $\mathcal{C}_i$ contains all the vertices of all levels of T of the form $L_{i+x(1+\lfloor \frac{r}{8} \rfloor)}$, where $x \in \mathbb{Z}_0^+$. By the pigeonhole principle, one of these classes will have at most $\lceil \frac{\alpha}{1+\lfloor \frac{r}{8} \rfloor} \rceil$ vertices. Again, by Lemma 5, given any vertex class $\mathcal{C}$ of T, all of $\mathcal{P}$ can be seen by the vertices of $\mathcal{C}$ when r diffuse reflections are allowed. The theorem follows.

Corollary 4 *The above bound (mentioned in Theorem 3) holds even if the guards are allowed to be placed anywhere on the boundary of the polygon.*

Proof. The proof follows directly from the proof of Theorem 3 since Lemmas 5 and 6 are valid for boundary guards as well.

Observation 2 *The above bound (mentioned in Theorem 3) does not hold in the case of arbitrary point guards.*

Proof. See the full version of the paper [32] for the proof.

Finding an approximate solution to the vertex guard problem with r diffuse reflections is harder than approximating the standard problem. Reflection may change the position of guards remarkably. Here, we have a straight-forward generalization of Ghosh's discretization algorithm presented in [33].

Theorem 5. *For vertex guards, the art gallery problem considering r reflections, for both the diffuse and specular reflection are solvable in $\mathcal{O}(n^{8^{r+1}+10})$ time giving an approximation ratio of $\mathcal{O}(\log n)$.*

Proof. See the full version of the paper [32] for the proof.

5 Conclusion

In this paper, we deal with a variant of the Art Gallery problem in which the guards are empowered with reflecting edges. Many applications consider one source and they want to make that source visible via various access points. Consider a WiFi network in an organization where due to some policies all the personnel should be connected to one specific network. The access points should receive the signal from one source and deliver it to places where the source cannot access. The problem is to minimizing the access points.

The gallery is denoted by a given simple polygon $\mathcal{P}$. This article mentioned a few versions of the problem of adding an area to the visibility polygon of a given point guard inside $\mathcal{P}$ as a viewer. Although we know that reflection could be helpful, we proved that several versions of the problem are *NP*-hard or *NP*-complete.

Nonetheless, we proved that although specular reflection might not help decrease the *minimum* number of guards required for guarding a gallery, diffuse reflection can decrease the optimal number of guards.

References

1. A. Vaezi, B. Roy, and M. Ghodsi., "Reflection helps guarding an art gallery.," *The* 38^{th} *European Workshop on Computational Geometry, Mar 2022, Perugia, Italy. hal-03674221*, pp. 3:1–3:7, 2022.
2. L. J. Guibas, J. Hershberger, D. Leven, M. Sharir, and R. E. Tarjan., "Linear-time algorithms for visibility and shortest path problems inside triangulated simple polygons.," *Algorithmica*, no. 2, pp. 209–233, 1987.
3. D. T. Lee., "Visibility of a simple polygon.," *Computer Vision, Graphics, and Image Processing*, no. 22, pp. 207—-221, 1983.
4. V. Klee., "Is every polygonal region illuminable from some point?," *Computational Geometry: Amer.Math. Monthly*, vol. 76, p. 180, 1969.
5. G. T. Tokarsky., "Polygonal rooms not illuminable from every point.," *American Mathematical Monthly.*, vol. 102, pp. 867–879, 1995.
6. B. Aronov, A. R. Davis, T. K. Dey, S. P. Pal, and D. Prasad., "Visibility with one reflection.," *Discrete & computational Geometry.*, no. 19, pp. 553–574, 1998.
7. B. A. A. R. Davis, T. K. Dey, S. P. Pal, and D. Prasad., "Visibility with multiple specular reflections.," *Discrete & computational Geometry.*, no. 20, pp. 62–78, 1998.
8. J. Urrutia., *Handbook of Computational Geometry.* 2000.
9. J. O'Rourke and K. Supowit., "Some np-hard polygon decomposition problems.," *IEEE Trans. Inf. Theory*, no. 29(2), pp. 181—-189, 1983.
10. D. T. Lee and A. Lin., "Computational complexity of art gallery problems.," *IEEE Transactions on Information Theory*, no. 32, pp. 276–282, 1986.
11. A. Aggarwal., "The art gallery theorem: its variations, applications and algorithmic aspects.," *Ph.D. thesis, The Johns Hopkins University, Baltimore, MD*, 1984.
12. M. Abrahamsen, A. Adamaszek, and T. Miltzow., "The art gallery problem is $\exists\mathbb{R}$-complete.," *Proceedings of the 50th Annual ACM SIGACT Symposium on Theory of Computing, STOC*, 2018.
13. M. Abrahamsen, A. Adamaszek, and T. Miltzow, "Irrational guards are sometimes needed," 2017.

14. S. K. Ghosh, "Approximation algorithms for art gallery problems.," *Proc. Canadian Information Processing Society Congress*, pp. 429–436, 1987.
15. J. King and D. Kirkpatrick., "Improved approximation for guarding simple galleries from the perimeter.," *Discrete Computer Geometry*, no. 46, pp. 252–269, 2011.
16. S. Ashur, O. Filtser, and M. J. Katz, "A constant-factor approximation algorithm for vertex guarding a wv-polygon," *J. Comput. Geom.*, vol. 12, no. 1, pp. 128–144, 2021.
17. A. Vaezi, "A constant-factor approximation algorithm for point guarding an art gallery," *arXiv:2112.01104*, 2021.
18. R. Guenther., "Modern optics.," *Wiley, New York*, 1990.
19. M. Born and E. Wolf., "Principles of optics.," *6th edn. Pergamon Press, Oxford*, 1980.
20. I. Newton., "Opticks, or a treatise of the reflections, refractions, inflections and colours of light,," *4th edn. London*, 1730.
21. C. Boldrighini, M. Keane, , and F. Marchetti., "Billiards in polygons.," *Ann.Probab.*, no. 6, pp. 532—-540, 1978.
22. E. Gutkin., "Billiards in polygons.," *Phys. D*, vol. 19, pp. 311—-333, 1986.
23. S. Kerckhoff, H. Masur, and J. Smillie., "Ergodicity of billiard flows and quadratic differentials. .," *Ann. of Math.*, vol. 124, p. 293–311, 1986.
24. V. V. Kozlov and D. V. Treshchev., "Billiards: A genetic introduction to the dynamics of systems with impacts.," *Translations of Mathematical Monographs, American Mathematical Society, Providence, RI*, vol. 89, pp. 62–78, 1991.
25. C. Xu., "A generalization of the art gallery theorem with reflection and a cool problem.," *https://chaoxuprime.com/posts/2011-06-06-a-generalization-of-the-art-gallery-theorem-with-reflection-and-a-cool-problem.html*, 2011.
26. G. Barequet, S. M. Cannon, E. Fox-Epstein, B. Hescott, D. L. Souvaine, C. D. Tóth, and A. Winslow, "Diffuse reflection diameter in simple polygons," *Discrete Applied Mathematics*, vol. 210, pp. 123–132, 2016. LAGOS'13: Seventh Latin-American Algorithms, Graphs, and Optimization Symposium, Playa del Carmen, México — 2013.
27. E. Fox-Epstein, C. D. Tóth, and A. Winslow, "Diffuse reflection radius in a simple polygon," *Algorithmica*, vol. 76, pp. 910–931, 2016.
28. A. Vaezi and M. Ghodsi, "Visibility extension via reflection-edges to cover invisible segments.," *Theoretical Computer Science*, 2019.
29. A. Vaezi and M. Ghodsi, "How to extend visibility polygons by mirrors to cover invisible segments.," *International Workshop on Algorithms and Computation*, pp. 42—53, 2017.
30. A. Vaezi and M. Ghodsi, "Extending visibility polygons by mirrors to cover specific targets," *EuroCG2013*, pp. 13—16, 2013.
31. J. Kleinberg and E. Tardos, *Algorithm design.* Pearson Education India, 2006.
32. A. Vaezi, B. Roy, and M. Ghodsi, "Visibility extension via reflection," 2020.
33. S. K. Ghosh., "Approximation algorithms for art gallery problems in polygons.," *Discrete Applied Mathematics*, no. 158(6), pp. 718–722, 2010.