Incentive Mechanism Design for Unicast Service Provisioning With Network Aggregative Game

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***Abstract*—We investigate the problem of distributed resource allocation in unicast communication networks with strategic/selfish users. First, through mechanism design, the centralized problem is converted to a decentralized problem that induces a network aggregative game among users. At every Nash equilibrium, this mechanism strongly implements the solution of the resource allocation problem, and it is budget balance as well. Then, we establish a relationship between Nash equilibria of the induced game and the solutions of the corresponding variational inequality problem. Next, a distributed algorithm is proposed, and finally, its converges to the Nash equilibrium of the induced game is proved under certain assumptions. Since each link can be shared among a different set of users, there is a specific connectivity graph among the users of each link. Hence, a user simultaneously utilizes multiple connectivity graphs to interact with different sets of neighbors on different links.**

***Index Terms*—Mechanism design, Nash-seeking algorithm, network aggregative game (NAG), variational inequality (VI).**

I. Introduction

Efficient allocation of network resources is of vital importance, especially when the resources are scarce and there are increasing demands for good-quality services.

Mechanism design, or reverse game theory, is a resource allocation approach [1], [2] that allows recovering a socially optimal solution in the presence of strategic users, through the design of interaction and allocation rules. The interaction rule determines how users should communicate with each other. The allocation rule specifies 1) how resources should be assigned and 2) how much users should be charged based on their interactions with others. These rules, along with the users' individual preferences, induce a game among users [1]. The network manager's goal is to design the interaction and allocation rules so that users' behaviors at the equilibrium of the induced game yield the optimal solution of the resource allocation problem [2]. The first attempts in this direction were based on the idea of the Vickery–Clarke–Groves (VCG) mechanism [3], [4].

Furthermore, the mechanism is desired to satisfy other criteria, such as budget balance and individual rationality [1]. The VCG and VCG-based mechanisms suffer from the lack of budget balance property. Mechanisms that are budget balance do not waste money [2]. They only use money to incentivize users not making a profit. This property is important from the viewpoints of both the network manager and the participants. If a mechanism runs into a deficit, the network manager needs to inject money into the system and, hence, is not favorable. In a mechanism that leads to a budget surplus, some amount of money will be left unused. There are several mechanisms in the literature that partially [4], [5] or fully [6], [7] satisfy these properties at equilibrium. However, these mechanisms do not provide an algorithm for the users with incomplete information to find their equilibrium strategies. A few papers in the literature provide an algorithm for their mechanism to reach the equilibrium point [8], [9]. However, they either miss some of the desired properties of a mechanism [8] or put some very restrictive assumptions on the valuation function and its derivatives [9].

Kakhbod and Teneketzis [6], [10] designed a mechanism for the unicast service-provisioning problem with strategic users that
1) implements the efficient allocation in all Nash equilibria,
2) is budget balanced,
3) is individually rational.

Nevertheless, the authors do not provide any numerical algorithm for finding the Nash equilibria, and to the best of the authors' knowledge, there is no Nash equilibrium (NE)-seeking algorithm for this mechanism yet.

From the computational viewpoint, network aggregative games (NAGs) have received much attention in recent years [11]–[13]. In NAGs, the neighbors to a user in the network affect the user's payoff function through an aggregative term, which is an aggregation of the decision values of all the user's neighbors. Therefore, each user only needs to know the aggregate decision values of its neighbors, and hence, the users' individual information privacy is preserved. Besides, the aggregative form of the games is more computationally efficient and has many interesting applications [14]–[17]. Nevertheless, NAGs have not been explored in a market-based mechanism design framework while simultaneously satisfying all the abovementioned desired features of the mechanism.

In this article, we investigate the problem of rate allocation in a unicast network with strategic users [6]. We assume that the network manager is not a profit maker, and users' valuation functions are their private information, which they are not willing to disclose. We propose a modified version of the mechanism proposed by Kakhbod and Teneketzis [6], [10] in which, while satisfying the desired properties like strong implementation, budget balance, and individual rationality, it induces an NAG among the agents. These are achieved mainly due to designing a new structure for the payment function. In addition, the

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induced NAG by our mechanism is an extended version of conventional NAGs [12], [13], [18], since the connectivity graph of the users changes for different links of a transmission network. This is due to the fact that each link can be shared among a different set of users. We show that by properly tuning mechanism parameters (i.e., payment function), the users' utility functions become concave. Hence, we use variational inequality (VI) theory to find an NE of the induced NAG [19]–[22]. Finally, we prove that the proposed Nash-seeking algorithm converges to an NE of the induced multigraph NAG.

Notations: $\mathbf{1}_q$ is a vector with q entries all equal to 1. $|x|$ and $\|x\|$ are the absolute value and Euclidean norm of x, respectively. Δ is the difference operator. $z^{lm} = (z^l)^m$ and $\mathbf{col}(s_1, s_2, \ldots, s_N) = (s_1^T, s_2^T, \ldots, s_N^T)^T$. The Euclidean projection of a vector x into a set $\mathcal{Y}$ is defined as $Proj_{\mathcal{Y}}^{(x)} \triangleq \text{argmin}_{y\in\mathcal{Y}}\|x - y\|$. $\bar{\mathcal{S}} \triangleq \sum_{i=1}^N \mathcal{S}_i = \{\bar{s} : \bar{s} = \sum_{j=1}^N s_j, s_1 \in \mathcal{S}_1, \ldots, s_N \in \mathcal{S}_N\}$. Given a closed convex set $\mathcal{K} \subset \mathbb{R}^n$ and a continuous function $H : \mathcal{K} \to \mathbb{R}^n$, $VI(\mathcal{K}, H)$ means finding a vector $\hat{z} \in \mathcal{K}$ such that $\langle H(\hat{z}), z - \hat{z}\rangle \geq 0 \forall z \in \mathcal{K}$.

II. Problem Formulation

Consider a wired transmission network that consists of L links and $N (N > 3)$ strategic/selfish users. $\mathbf{L} = \{l_1, \ldots, l_L\}$ and $\mathbf{I} = \{1, \ldots, N\}$, respectively, denote the set of network links and users. Also, the capacity of link $l \in \mathbf{L}$ is limited by c_l. The network topology is fixed, and each user exploits one fixed, predetermined route to send information from a source to a destination. A route consists of a sequence of links starting from a source node to a destination node. User i's transmission route is shown by $R_i \subseteq \mathbf{L}$, and $l_1^i, \ldots, l_{q_i}^i$ is its corresponding sequence of links. Here, $l_r^i \in R_i$ shows the rth link of user i's route, and q_i is the number of links establishing R_i. For each $l \in \mathbf{L}$, we denote the set and number of users that are using link l in their routes by g^l and N_l, respectively.

User i transmits data over all links of his route R_i with a rate $x_i \geq 0$. The satisfaction of user i from transmitting data at rate x_i is expressed by a valuation function $V_i(x_i)$.

Assumption 1: For each $i \in \mathbf{I}$, The valuation function $V_i(x_i) : \mathcal{X}_i \subseteq [0, \min_{l\in R_i} c_l] \to \mathbb{R}_{\geq 0}$ is an increasing concave function of x_i with $V_i(0) = 0$. This function, which is continuously differentiable in x_i, is the private information of user i.

The network manager's goal is to maximize the network welfare function W while satisfying network constraints, i.e.,

$$\begin{aligned} \max_{x}.\quad & W(x) = \sum_{i=1}^N V_i(x_i) \\ \text{s.t.} & \sum_{i\in g^l} x_i \leq c_l \quad \forall l \in \mathbf{L} \\ & x_i \geq 0, \quad i \in \mathbf{I} \end{aligned} \tag{1}$$

where $x = (x_1, \ldots, x_N)^T$ is called the (resource) allocation vector, and the first constraint is due to the limited resource on each link. Optimization (1) would be solved if the network manager knew all the valuation functions $V_i(x_i)\ \forall i \in \mathbf{I}$. In addition, since users are strategic/selfish, an individual user may have incentive to change its strategy from the optimizer of (1) to increase its profit. In Section III, we propose an incentive mechanism to overcome these issues.

III. Monetary Incentive Mechanism for Rate Allocation

A monetary incentive mechanism, together with the users' utility functions, induces a game among users. The interaction of the strategic users in this game determines the values of allocated resources. In this mechanism, each user i is asked to send a message s_i chosen from his convex, compact strategy space $\mathcal{S}_i \subset \mathbb{R}_{\geq 0}^{q_i+1}$ to the network manager. User i's message is a vector of the form $s_i = (x_i, p_i^{l_1^i}, p_i^{l_2^i}, \ldots, p_i^{l_{q_i}^i})^T$. Here, $x_i \in \mathcal{X}_i \subseteq [0, \min_{l\in R_i} c_l]$ indicates the rate requested by user i, and $p_i^{l_r^i} \in \mathcal{P}^{l_r^i} \subset \mathbb{R}_{\geq 0}$ is user i's proposed price for using link $l_r^i \in R_i$. The message space of the mechanism is defined as $\mathcal{S} = \prod_{i=1}^N \mathcal{S}_i$. The outcome function $f : \mathcal{S} \to \mathbb{R}_{\geq 0}^N \times \mathbb{R}^N$ of the mechanism determines the outcome $f(s)$ for any given message profile $s = \mathbf{col}(s_1, s_2, \ldots, s_N) \in \mathcal{S}$. The outcome function of the mechanism is of the form $f(s) = (y(s), t(s))$, where the allocation function $y(.) = (y_1(.), y_2(.), \ldots, y_N(.))^T$ determines the transmission rates, and the payment function $t(.) = (t_1(.), t_2(.), \ldots, t_N(.))^T$ determines the monetary payments the users must make in the mechanism.

We design our mechanism such that the manager assigns the same amount of resources requested by each user, i.e., $y_i(s) = x_i$. Hereafter, we write x and x_i instead of $y(s) = x$ and $y_i(s) = x_i$, respectively. Also, we define payment $t_i(s)$ to be paid by each user i as follows:

$$t_i(s) = \sum_{l\in R_i} t_i^l(s) \tag{2}$$

where t_i^l is user i's payment for using link l in his route. The function t_i^l is defined as follows:

$$\begin{aligned} t_i^l = & \frac{N_l}{(N_l - 1)^2}(-p_i^l + \bar{P}^l)\left(x_i - \frac{\bar{X}^l}{N_l}\right) \\ & + \left[\frac{N_l}{N_l - 1}\left(p_i^l - \frac{\bar{P}^l}{N_l}\right) - \zeta_+^{l2}\right]^2 \\ & - \frac{2}{\gamma}\cdot\frac{N_l}{(N_l - 1)^2}(-p_i^l + \bar{P}^l)\left(p_i^l - \frac{\bar{P}^l}{N_l}\right)(\bar{X}^l - c_l) \end{aligned} \tag{3}$$

where

$$\bar{X}^l = \sum_{j\in g^l} x_j \in \bar{\mathcal{X}}^l = \sum_{j\in g^l} \mathcal{X}_j,\ \bar{P}^l = \sum_{j\in g^l} p_j^l \in \bar{\mathcal{P}}^l = \sum_{j\in g^l} \mathcal{P}_j^l \tag{4}$$

$$\zeta_+^l = \max\left\{0, \frac{\bar{X}^l - c_l}{\hat{\gamma}}\right\},\ \zeta_+^{l2} = (\zeta_+^l)^2. \tag{5}$$

Parameters γ and $\hat{\gamma}$ are positive constants that should be appropriately chosen by the network manager.

The payment of user i defined by (2) and (3) can be positive or negative. The total utility of user i who is allowed to transmit with rate x_i in exchange for making payment t_i is given as follows:

$$U_i(s) = V_i(x_i) - t_i(s) = V_i(x_i) - \sum_{l\in R_i} t_i^l(s). \tag{6}$$

We can see from (3) that for each $l \in R_i$, tax function $t_i^l(s)$ depends on user i's decision variables x_i and p_i^l, and the aggregate of the decision variables of the users who are using link l, i.e., $\bar{X}^l$ and $\bar{P}^l$. Therefore, this mechanism induces an aggregative game on every link of the network among competing users. On the other hand, these games are coupled to each other through the allocation vector x. To make this point more clear, hereafter, we indicate the utility of user i in this NAG by $U_i(s_i, \bar{X}_i, \bar{P}_i)$, where $\bar{X}_i = (\bar{X}^{l_1^i}, \ldots, \bar{X}^{l_{q_i}^i})^T$ and $\bar{P}_i = (\bar{P}^{l_1^i}, \ldots, \bar{P}^{l_{q_i}^i})^T$ are the aggregates of rates and price levels announced by the users who are using the links $l_1^i, \ldots, l_{q_i}^i$. Also, note that $\bar{X}_i \in \bar{\mathcal{X}}_i = \prod_{r=1}^{q_i} \bar{\mathcal{X}}^{l_r^i}$ and $\bar{P}_i \in \bar{\mathcal{P}}_i = \prod_{r=1}^{q_i} \bar{\mathcal{P}}^{l_r^i}$.

IV. Properties of the Proposed Mechanism

The mechanism proposed in Section III together with any set of valuation functions $V_i(x_i)$, $i \in \mathbf{I}$, induces an NAG $\mathcal{G}$ among users, where $\mathcal{G} := (\mathbf{I}, (\mathcal{S}_i)_{i\in\mathbf{I}}, (U_i(s_i, \bar{X}_i, \bar{P}_i))_{i\in\mathbf{I}})$. Agent i's information set is $\mathscr{I}_i = \{\gamma, \hat{\gamma}, N_l, c_l, l \in R_i\}$. Each user i chooses his strategy s_i so as to maximize his utility function $U_i(s_i, \bar{X}_i, \bar{P}_i)$. We study the induced games by considering NE as a solution concept.

Definition 1 (NE [2]): The strategy profile $s^* = \mathbf{col}(s_1^*, \ldots, s_N^*) \in \mathcal{S}$ is an NE of the game $\mathcal{G}$, if

$$U_i(s_i^*, \bar{X}_i^*, \bar{P}_i^*) \geq U_i\left(s_i, \bar{X}_i^* - x_i^*.\mathbf{1}_{q_i} + x_i.\mathbf{1}_{q_i}, \bar{P}_i^* - p_i^* + p_i\right)$$

for all $i \in \mathbf{I}$ and $s = \mathbf{col}(s_1, \ldots, s_N) \in \mathcal{S}$, where $p_i = (p_i^{l_1^i}, p_i^{l_2^i}, \ldots, p_i^{l_{q_i}^i})^T$ is the vector of user i's proposed prices. In a word, in an NE, no user could be able to increase his utility by unilaterally deviating from his strategy.

Proposition 1: Suppose $s^* = \mathbf{col}(s_1^*, s_2^*, \ldots, s_N^*)$ is an NE of the game induced by the mechanism. Then, we have the following relationships for all $l \in \mathbf{L}$ and $i, j \in \mathbf{I}$:

$$\zeta_+^{l*} = 0, \qquad p_i^{l*} = p_j^{l*} = \frac{\bar{P}^{l*}}{N_l} = p^{l*} \tag{7}$$

$$p^{l*}(\bar{X}^{l*} - c_l) = 0, \qquad \left.\frac{\partial t_i^l}{\partial x_i}\right|_{s=s^*} = p^{l*}. \tag{8}$$

Proposition 1 guarantees the feasibility of allocation at all Nash equilibria [(7)-left]. Also, it shows users competing on each link reach an agreement on the proposed price for that link [(7)-right], and they do not have to pay money for using a link which is not fully utilized [(8)-left].

Using the results of Proposition 1, we have the following important theorem.

Theorem 1: The proposed mechanism has the following properties at every NE, s^*, of the game induced by the mechanism.

i) *Nash existence:* Assume that optimization problem (1) has a solution x^{**}. Then, there is a Nash strategy profile s^* which results in $x(s^*) = x^{**}$.

ii) *Strong implementation:* Resulting resource allocation is a solution of the optimization problem (1).

iii) *Budget balance:* Total payment of agents on the whole network as well as every link are equal to zero, i.e., $\sum_{i=1}^N t_i(s^*) = 0$, $\sum_{i\in g^l} t_i^l(s^*) = 0$.

iv) *Individual rationality:* Users prefer the outcome of every NE to the outcome of not participating.

V. Solving Nash-Seeking Problem Via VI Theory

In this section, we address the question that how could users with incomplete information reach an NE? We answer this question through VI theory. Before continuing, some assumptions should be made.

Assumption 2: Function $-V_i(x_i)$ $\forall i \in \mathbf{I}$, is α_i strongly convex and has bounded first derivative on the interval $[0, \min_{l\in R_i} c_l]$. The coefficients α_i and upper bounds are known to the network manager.

Assumption 3: User i's (individual) strategy space is a hypercube which encompasses his strategy at NE, i.e.,

$$\mathcal{S}_i \equiv \left\{ s_i \in \mathbb{R}_{\geq 0}^{q_i+1} : x_i \leq \min_{l\in R_i} c_l,\ p_i^l \leq P_{\max}^l,\ l \in R_i \right\}. \tag{9}$$

Remark 1: According to Proposition 1, Theorem 1, and based on the KKT conditions of (1), we have $p^{l*} \leq \frac{\partial V_i(x_i^*)}{\partial x_i} \forall i \in g^l$. Therefore, based on Assumption 2, the network manager can determine sufficiently large $P_{\max}^l \forall l \in \mathbf{L}$, such that the NE is not confined.

Proposition 2: Suppose that Assumptions 1–3 hold. Then, parameters γ and $\hat{\gamma}$ in (2)–(5) can be chosen such that user i's utility function U_i becomes concave in his strategy s_i.

Now, we construct a VI such that its solutions coincide with the NE of the games induced by our mechanism. To this end, we define the operator $F : \mathcal{S}_F \times \bar{\mathcal{X}}_F \times \bar{\mathcal{P}}_F \to \mathbb{R}^{N+\sum_{i=1}^N q_i}$ as follows:

$$F_i(s_i, \bar{X}_i, \bar{P}_i) = -\nabla_{s_i} U_i(s_i, \bar{X}_i, \bar{P}_i) \tag{10}$$

$$F(s, \bar{X}, \bar{P}) := \mathbf{col}(F_1, F_2, \ldots, F_N) \tag{11}$$

where $\mathcal{S}_F = \prod_{i=1}^N \mathcal{S}_i$, $\bar{\mathcal{X}}_F = \prod_{r=1}^L \bar{\mathcal{X}}^{l_r}$, $\bar{\mathcal{P}}_F = \prod_{r=1}^L \bar{\mathcal{P}}^{l_r}$. Now, we define the operator Ψ as follows:

$$\Psi(s) \triangleq F(s, \bar{X}, \bar{P}). \tag{12}$$

Operator Ψ is well defined over the set $\mathcal{S}_F$. The following proposition establishes a connection between solutions of $VI(\mathcal{S}_F, \Psi(s))$ and Nash equilibria of game $\mathcal{G}$.

Proposition 3 ([19], Proposition 1.4.2): Suppose γ and $\hat{\gamma}$ are chosen such that Proposition 2 is satisfied. Strategy profile $s^* = \mathbf{col}(s_1^*, \ldots, s_N^*)$ is an NE of the game $\mathcal{G}$ if and only if it solves $VI(\mathcal{S}_F, \Psi)$.

VI. Distributed Algorithm

This section introduces a distributed learning algorithm to find a solution of $VI(\mathcal{S}_F, \Psi)$. Contrary to the standard distributed algorithms [23] that utilize only one connectivity graph for all users, here, each user i with q_i links in his route R_i uses q_i different connectivity graphs to share information with (possibly) distinguished sets of his neighbors.

We show the connectivity graph of each link at iteration k by $\mathscr{G}^l = (g^l, \varepsilon^l(k), W^l(k))$, where g^l and $\varepsilon^l(k)$ are the sets of nodes and edges, respectively, and $W^l(k) = [w_{ij}^l(k)]$ is the weight matrix, determined by the manager, representing the weights users assign to the information they receive from their neighbors.[1] Also, $\mathcal{N}_i^l$ indicates the set of user i's neighbors on link l. We assume that the connectivity graphs satisfy the following conditions.

Assumption 4: All connectivity graphs are undirected.

Assumption 5: For each link l, there is an integer $B^l \geq 1$ which makes the graph $G^{l,B^l} = (g^l, \bigcup_{r=1}^{B^l} \varepsilon^l(k+r))$ connected for all $k \geq 0$.

Assumption 6: Each weight matrix $W^l(k)$ is doubly stochastic, i.e., $\sum_i w_{ij}^l = 1$ and $\sum_j w_{ij}^l = 1$. Moreover, $w_{ij}^l(k) \geq \delta^l > 0$ for all $j \in \mathcal{N}_i^l(k)$ and $w_{ij}^l(k) = 0$ for $j \notin \mathcal{N}_i^l(k)$.

Next, the convergence of Algorithm 1 is studied.

Proposition 4: Suppose Assumptions 1–3 hold. There are proper choices of γ and $\hat{\gamma}$, which make the operator $F(s, \bar{X}, \bar{P})$ strictly monotone in its first argument s with fixed $\bar{X}$ and $\bar{P}$ over the compact and convex set $\mathcal{S}_F$ if the strong convexity coefficients α_i, $i \in \mathbf{I}$, satisfy $\alpha_i > 1/2 \sum_{l\in R_i} 1/(N_l - 1)$.

Remark 2: The condition $\alpha_i > 1/2 \sum_{l\in R_i} 1/(N_l - 1)$ can be always satisfied if the manager multiplies the objective function (1) by a sufficiently large positive scalar β. In this case, the optimal allocation remains unchanged, but optimal prices will be $p_{\text{new}}^{l*} = \beta p^{l*}$ (see Proposition 1).

Corollary 1: Suppose $\gamma, \hat{\gamma}$, and $V_i(x_i)$, $i \in \mathbf{I}$ are such that Proposition 4 holds. Since F is continuous and strictly monotone on compact and convex set $\mathcal{S}_F$, $VI(\mathcal{S}_F, F)$ with fixed $\bar{X}$ and $\bar{P}$ has a unique solution on $\mathcal{S}_F$ [19]. As $\bar{X}$ and $\bar{P}$ converge to $\bar{X}^*$ and $\bar{P}^*$, respectively, then this solution is also a solution of $VI(\mathcal{S}_F, \Psi)$.

[1] The weight matrix is known to the users. We capture this feature by modifying each user i's information set as $\mathscr{I}_i = \{\gamma, \hat{\gamma}, \eta_k, N_l, c_l, W^l, l \in R_i\}$.

Algorithm 1: Distributed Resource Allocation.

Initialization

-Network Manager:

set $\gamma, \hat{\gamma}, \eta_k, c_l, P^l_{\text{max}}, W^l \; \forall l \in \mathbf{L}$.

-All users $i \in \mathbf{I}$**:**

$k = 0, \quad s_i(0) \in \mathcal{S}_i,$

$\tilde{X}^l_i(0) = x_i(0), \quad \tilde{P}^l_i(0) = p^l_i(0) \quad \forall l \in R_i$

Each user $i = 1, \ldots, N \; \forall l \in R_i,$

Do Until Convergence:

-Updating local estimations: Each user i updates his estimation of the aggregative terms for all links of his route

$$\hat{X}^l_i(k) = \textstyle\sum_{j \in g^l} w^l_{ij}(k) \tilde{X}^l_j(k)$$
$$\hat{X}_i(k) = (N_{l^i_1} \hat{X}^{l^i_1}_i(k) \; \cdots \; N_{l^i_{q_i}} \hat{X}^{l^i_{q_i}}_i(k))^T$$
$$\hat{P}^l_i(k) = \textstyle\sum_{j \in g^l} w^l_{ij}(k) \tilde{P}^l_j(k)$$
$$\hat{P}_i(k) = (N_{l^i_1} \hat{P}^{l^i_1}_i(k) \; \cdots \; N_{l^i_{q_i}} \hat{P}^{l^i_{q_i}}_i(k))^T$$

-Updating the decision variable

$$s_i(k+1) = \underset{\mathcal{S}_i}{\text{Proj}}[s_i(k) - \eta_k F_i(s_i(k), \hat{X}_i(k), \hat{P}_i(k))]$$

-Correction of the local estimations

$$\tilde{X}^l_i(k+1) = \hat{X}^l_i(k) + x_i(k+1) - x_i(k)$$
$$\tilde{P}^l_i(k+1) = \hat{P}^l_i(k) + p^l_i(k+1) - p^l_i(k)$$

end

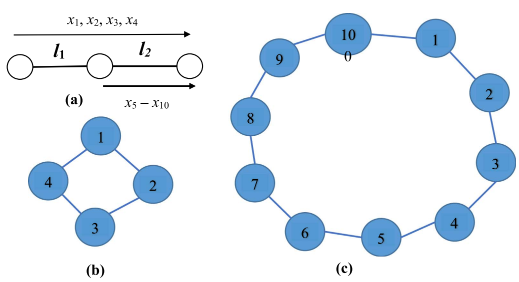

Fig. 1. (a) Two-link transmission network. Connectivity graph for users of (b) link 1, l_1 and (c) link 2, l_2.

Proposition 5: Under Assumptions 4–6, Propositions (2), (4), and (6), and Lemmas (2) and (3), Algorithm 1 converges to a solution of $VI(\mathcal{S}_F, \Psi)$, if the learning rate η_k is monotonically nonincreasing and satisfies $\sum_{k=1}^{\infty} \eta_k = \infty, \sum_{k=1}^{\infty} \eta_k^2 < \infty$.

VII. Numerical Simulations

Consider a network that consists of two links and ten users [see Fig. 1(a)]. The first four users transmit data over both links while the others only use the second link. The links capacities are $c_{l_1} = 4$ and $c_{l_2} = 18$. The user i's valuation function is of the form $V_i(x_i) = d_i\sqrt{x_i}$, where d_i is a positive constant. d_i's are considered to be $[d_1 \; \cdots \; d_{10}] = [2\;1\;1\;2\;1\;3\;1\;2\;2\;3]$, and in this case, the optimal solution is $x^* = [1.6\;0.4\;0.4\;1.6\;0.5\;4.5\;0.5\;2\;2\;4.5]^T$. Fig. 1(b) and (c) shows the connectivity graphs among users of each link. In Fig. 1(b) and (c), a blue numbered circle represents a user. If two circles are connected, it means that those two users are neighbors on that link. In these examples for both weight matrices, $w^l_{ii} = 0.5$, and $w^l_{ij} = 0.25$ if i and j are neighbors, otherwise $w^l_{ij} = 0$. Other algorithm parameters are $\gamma = 20$, $\hat{\gamma} = 2$, $P^{l_1}_{\text{max}} = 4$, $P^{l_2}_{\text{max}} = 10$, and here, the learning rate is constant and $\eta_k = 0.15$.

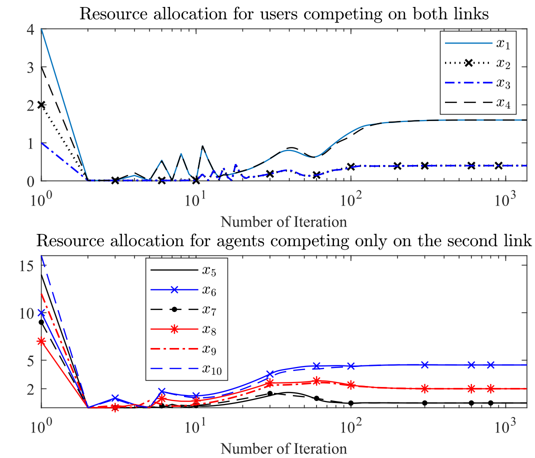

Fig. 2. Resource allocation in each iteration.

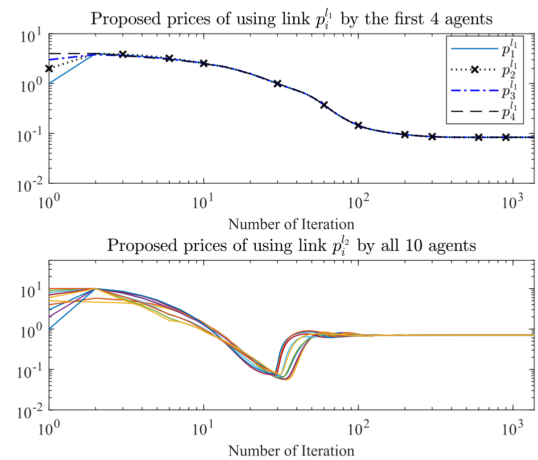

Fig. 3. Proposed prices p^{l_1} and p^{l_2} by users in each iteration.

After running the algorithm for 1371 iterations, it converges to the optimal solution, x^* (see Fig. 2), $p^{l_1 *} = 0.0835$, and $p^{l_2 *} = 0.7071$.

Also, users reach consensus on the prices very fast on both links (see Fig. 3). Both Figs. 2 and 3 are in accordance with our intuition about the market and resources. When a resource becomes scarce or, equivalently, is demanded by too many customers, the price of that resource increases, and when that resource exists in abundance, its price will reduce.

VIII. Conclusion

In this article, the problem of resource allocation in unicast networks has been investigated. First, the problem through mechanism design converted to an NAG between strategic users, and then, to find the Nash point of the game, VI theory has been used. Furthermore, to protect the users' privacy and remove the computational load in one node, the algorithm runs in a distributed manner.

Appendix A
Proof of the Propositions

Let us first provide a detailed version of operator F_i

$$F_i(s_i, \bar{X}_i, \bar{P}_i) = \begin{pmatrix} F_{ix} \; F_{ip^{l^i_1}_i} \; \ldots \; F_{ip^{l^i_{q_i}}_i} \end{pmatrix}^T \tag{13}$$

$$F_{ix} = -\frac{\partial U_i}{\partial x_i} = -\frac{dV_i(x_i)}{dx_i} + \sum_{l\in R_i}\Bigg\{\frac{1}{N_l-1}(-p_i^l+\bar{P}^l) - \frac{4}{\hat{\gamma}}\zeta_+^l\left(\frac{N_l}{N_l-1}p_i^l - \frac{1}{N_l-1}\bar{P}^l - \zeta_+^{l2}\right) - \frac{2}{\gamma}\cdot\frac{N_l}{(N_l-1)^2}\left(-p_i^l+\bar{P}^l\right)\left(p_i^l - \frac{\bar{P}^l}{N_l}\right)\Bigg\} \tag{14}$$

$$F_{ip_i^l} = -\frac{\partial U_i}{\partial p_i^l} = 2\left(\frac{N_l}{N_l-1}p_i^l - \frac{1}{N_l-1}\bar{P}^l - \zeta_+^{l2}\right) - \frac{2}{\gamma}\cdot\frac{1}{N_l-1}\left(-p_i^l+\bar{P}^l\right)\left(\bar{X}^l - c_l\right) \quad \forall l\in R_i. \tag{15}$$

Proof of Proposition 1: The Nash points are the solutions of N joint optimization problems of the kind

$$\max_{s_i}.\, U_i(s_i,\bar{X}_i,\bar{P}_i) = \min_{s_i}.\, -U_i(s_i,\bar{X}_i,\bar{P}_i)$$
$$\text{s.t.} -x_i \le 0 \ -p_i^l \le 0,\ l\in R_i$$

or equivalently, in terms of the Lagrangian of each user

$$\mathcal{L}_i = -U_i(s_i,\bar{X}_i,\bar{P}_i) - \kappa_i x_i - \sum_{l\in R_i}\nu_i^l p_i^l.$$

Now, let s^* be an NE. Hence, the KKT conditions imply that $\forall i\in\mathbf{I}$ and $\forall l\in R_i$

$$\begin{cases}\nabla_{s_i}\mathcal{L}_i\Big|_{s=s^*} = 0 \Rightarrow \begin{cases}\partial\mathcal{L}_i/\partial x_i\Big|_{s=s^*} = 0\\ \partial\mathcal{L}_i/\partial p_i^l\Big|_{s=s^*} = 0\end{cases}\\ \kappa_i^* x_i^* = 0, \quad \kappa_i^*\ge 0, \quad x_i^*\ge 0\\ \nu_i^{l*}p_i^{l*} = 0, \quad \nu_i^{l*}\ge 0, \quad p_i^*\ge 0.\end{cases} \tag{16}$$

Using (15), we have

$$\frac{\partial\mathcal{L}_i}{\partial p_i^l}\Big|_{s=s^*} = 0 \quad \forall i\in\mathbf{I} \quad \forall l\in R_i$$
$$\Rightarrow \left(\frac{N_l}{N_l-1}p_i^{l*} - \frac{1}{N_l-1}\bar{P}^{l*} - \zeta_+^{l*2}\right) - \frac{2}{\gamma}\cdot\frac{1}{N_l-1}\times (-p_i^{l*}+\bar{P}^{l*})(\bar{X}^{l*}-c_l) - \nu_i^{l*} = 0. \tag{17}$$

Equation (17) holds for all users using link l. So

$$\sum_{j\in g^l}\frac{\partial\mathcal{L}_j}{\partial p_j^l}\Big|_{s=s^*} = -N_l\zeta_+^{l*2} - \frac{2}{\gamma}\cdot\bar{P}^{l*}\left(\bar{X}^{l*}-c_l\right) - \sum_{j\in g^l}\nu_j^{l*} = 0 \tag{18}$$

if $\zeta_+^{l*}\ne 0$, then it contradicts (18). Hence $\zeta_+^{l*} = 0$. Now, following two cases could happen.

Case 1-a) $\bar{X}^{l*} - c_l = 0$. Equation (18) simply results in $\sum_{j\in g^l}\nu_j^{l*} = 0$, and consequently $\nu_j^{l*} = 0$ for $j\in g^l$. Thus, (17) results in

$$p_i^{l*} = \frac{\bar{P}^{l*}}{N_l} \quad \forall i\in g^l \Rightarrow p_i^{l*} = p_j^{l*} = p^{l*} \quad \forall i,j\in g^l. \tag{19}$$

Case 1-b) $\bar{X}^{l*} - c_l < 0$. If all $p_j^{l*} = 0 \forall j\in g^l$, then we reach the same conclusion of case (1-a). Hence, suppose that there is at least one $j\in g^l$ that $p_j^{l*}\ne 0$. We proceed with the proof by contradiction and show that $p_i^{l*} = p_j^{l*}\forall i,j\in g^l$ in this case too. Here, there is at least one user that $p_j^{l*} > \bar{P}^{l*}/N_l$. Now, according to (3) for t_j^l, user j can decrease his proposed price p_j^{l*}, while the others' are fixed, to reduce his cost and this contradicts being in a Nash point. So, we deduce that $p_i^{l*} = p_j^{l*} = p^{l*}\forall i,j\in g^l$, and if $p^{l*}\ne 0$, then it contradicts (18) again. Hence, $p^{l*} = 0$.

As it is evident, in both cases $p^{l*}(\bar{X}^{l*} - c_l) = 0$. Furthermore, $\frac{\partial t_i^l}{\partial x_i}\big|_{s=s^*} = p_i^{l*}$. ■

Proof of Theorem 1

i) *Proof of Property i)*: Lagrangian of the optimization (1) is given as follows:

$$\mathcal{L} = -\sum_{i=1}^{N}V_i(x_i) + \sum_{l\in\mathbf{L}}\lambda^l\left(\sum_{i\in g^l}x_i - c_l\right) - \sum_{i=1}^{N}\mu_i x_i. \tag{20}$$

Let $x^{**} = (x_1^{**},\ldots,x_N^{**})^T$, $\mu_i^{**} = (\mu_1^{**},\ldots,\mu_N^{**})$, and λ^{l**}, $l\in\mathbf{L}$, be a solution of optimization (1). $s^* = \mathbf{col}(s_1^*,\ldots,s_N^*)$ is an NE if (16) holds for s^*. Equation (16) holds by considering $x_i^* = x_i^{**}$, $\kappa_i^* = \mu_i^{**}$, $\nu_i^{l*} = 0\forall l\in\mathbf{L}$ (see the proof of Proposition 1), and $p^{l*} = \lambda^{l*}$.

ii) *Proof of Property ii):* Let $s^* = \mathbf{col}(s_1^*,\ldots,s_N^*)$ be an arbitrary NE. Then, allocation at the NE, $x^* = (x_1^*,x_2^*,\ldots,x_N^*)^T$, is a feasible solution of the optimization problem (1). Then, the optimal solution of problem (1) must satisfy KKT conditions for Lagrangian (20). Hence, $\forall i\in\mathbf{I}$ and $\forall l\in R_i$, we can write

$$\begin{cases}\frac{\partial V_i}{\partial x_i}(x_i^{**}) + \sum_{l\in R_i}\lambda^{l**} + \mu_i^{**} = 0\\ \mu_i^{**}x_i^{**} = 0, \quad x^{**}\ge 0, \quad \mu_i^{**}\ge 0\\ \lambda^{l**}(\sum_{i\in g^l}x_i^{**} - c_l) = 0\\ \sum_{i\in g^l}x_i^{**} - c_l \le 0, \quad \lambda^{l**}\ge 0\end{cases} \tag{21}$$

where the double star sign is used for the solution of the optimization (1). Comparing (21) with (7), (8), and (16) shows that by choosing $x_i^{**} = x_i^*$, $\mu_i^{**} = \kappa_i^*$, and $\lambda^{l**} = p^{l*}$, every NE is a solution of the optimization (1).

iii) *Proof of Property iii):*

$$\sum_{i\in g^l}t_i^{l*} = \sum_{i\in g^l}\frac{N_l}{N_l-1}p^{l*}\left(x_i^* - \frac{\bar{X}^{l*}}{N_l}\right) = 0$$

and since $\sum_{i\in\mathbf{I}}t_i^* = \sum_{i\in\mathbf{I}}\sum_{l\in R_i}t_i^{l*} = \sum_{l\in\mathbf{L}}\sum_{i\in g^l}t_i^{l*}$, the proposition is proved.

iv) *Proof of Property iv):* If user i does not use the network, he gets nothing, i.e., $U_i = 0$. So, it is sufficient to show that each user receives a nonnegative utility at every NE. This quantity is equal to

$$U_i(s_i^*,\bar{X}_i^*,\bar{P}_i^*) = V_i\left(x_i^*\right) - \sum_{l\in R_i}\frac{N_l}{N_l-1}p^{l*}\left(x_i^* - \frac{\bar{X}^{l*}}{N_l}\right). \tag{22}$$

Since it is an NE, no user can increase its utility by unilaterally deviating from his strategy. Consider the case when only user i deviates from his demand, it is concluded that

$$U_i\left(s_i^*,\bar{X}_i^*,\bar{P}_i^*\right) \ge U_i\left(s_i,\bar{X}_i,\bar{P}_i^*\right) = V_i(x_i) - \sum_{l\in R_i}p^{l*}x_i + \sum_{l\in R_i}\frac{1}{N_l-1}p^{l*}\bar{X}_{-i}^{l*}. \tag{23}$$

Here, $\bar{X}_{-i}^{l*} = \sum_{j\in g^l, j\ne i}x_j^*$. The right-hand side of the equality sign in (23) is a single-variable function of x_i. It is obvious that if $x_i = 0$, then $U_i\ge 0$. From Assumptions 1 and 2 and Proposition 1, we can conclude that this function has a maximum at x_i^*, and consequently, it proves that at the NE, $U_i(s_i^*,\bar{X}_i^*,\bar{P}_i^*)\ge 0$.

Lemma 1: Assume that $f:\mathcal{Z}\subseteq\mathbb{R}^n\to\mathbb{R}$ is α strongly convex. Then, the following relation is true for all $z_1,z_2\in\mathcal{Z}$:

$$\langle\Delta\nabla f(z),\Delta z\rangle \ge \alpha\|\Delta z\|^2. \tag{24}$$

Proof: Based on α strongly convexity of f, we have

$$f(z_1) \geq f(z_2) + \nabla f(z_2)^T (z_1 - z_2) + \frac{\alpha}{2}\|z_1 - z_2\|^2.$$

Adding the abovementioned inequality to the original one yields (24).

Proof of Proposition 2: First, we reformulate the payment function (3) to achieve new expression $U_i(s_i, \bar{X}_{-i}, \bar{P}_{-i})$

$$t_i^l = \frac{\bar{P}_{-i}^l}{N_l - 1}\left(x_i - \frac{\bar{X}_{-i}^l}{N_l - 1}\right) + \left(p_i^l - \frac{\bar{P}_{-i}^l}{N_l - 1} - \zeta_+^{l2}\right)^2 - \frac{2}{\gamma}\cdot\frac{\bar{P}_{-i}^l}{N_l - 1}\left(p_i^l - \frac{\bar{P}_{-i}^l}{N_l - 1}\right)(x_i + \bar{X}_{-i}^l - c_l)$$

where $\bar{P}_{-i}^l = \sum_{j\in g^l, j\neq i} p_j^l$, $\bar{X}_{-i}^l = \sum_{j\in g^l, j\neq i} x_j$, and ζ_+^l was defined in (5). We know that the function U_i is concave in s_i over the set $\mathcal{S}_i$ iff $-U_i$ be convex in s_i over $\mathcal{S}_i$ and vice versa. ([24], Th. 1.17)

$$\langle -\Delta_{s_i}\nabla_{s_i} U_i(s_i, , \bar{X}_{-i}, \bar{P}_{-i}), \Delta s_i\rangle \geq 0 \ \ i \in \mathbf{I}.$$
$$\Rightarrow -\Delta_{s_i}\frac{\partial U_i}{\partial x_i}\Delta x_i - \sum_{l\in R_i}\Delta_{s_i}\frac{\partial U_i}{\partial p_i^l}\Delta p_i^l \geq 0. \quad (25)$$

After some algebraic operations, the left-hand side (LHS) of the inequality (25) transforms to the following relation:

$$-\Delta\frac{dV_i}{dx_i}(x_i)\Delta x_i + \sum_{l\in R_i}\left\{-4\left(\frac{\zeta_+^l}{\hat{\gamma}} + \frac{\bar{P}_{-i}^l}{\gamma(N_l - 1)}\right)\Delta x_i \Delta p_i^l + \frac{4}{\hat{\gamma}}\left(-p_{i2}^l + \frac{\bar{P}_{-i}^l}{N_l - 1} + \zeta_{+1}^{l2} + \zeta_{+1}^l\zeta_{+2}^l + \zeta_+^{l2}\right)\Delta\zeta_+^l\Delta x_i - 2(\zeta_{+1}^l + \zeta_{+2}^l)\Delta\zeta_+^l\Delta p_i^l + 2(\Delta p_i^l)^2\right\}.$$

In writing the abovementioned equation, we rewrite the term $\Delta(\zeta_+^l p_i^l) = \zeta_{+2}^l p_{i2}^l - \zeta_{+1}^l p_{i1}^l$ into $p_{i2}^l\Delta\zeta_+^l + \zeta_{+1}^l\Delta p_i^l$. Considering the fact that $2y_1y_2 \leq (y_1^2 + y_2^2)$, and here, $|\Delta\zeta_+^l| \leq |\Delta x_i|/\hat{\gamma}$ as well as Assumption 2 and Lemma 1, and also avoiding the term $(\frac{\bar{P}_{-i}^l}{N_l-1} + \zeta_{+1}^{l2} + \zeta_{+1}^l\zeta_{+2}^l + \zeta_{+2}^{l2})\Delta\zeta_+^l\Delta x_i$ as it is always nonnegative, the preceding expression has a lower bound expressed as follows:

$$\text{LHS} \geq (\alpha_i - 2M_{ix})(\Delta x_i)^2 + 2\sum_{l\in R_i}\left(1 - M_{ip}^l\right)\left(\Delta p_i^l\right)^2 \quad (26)$$

where

$$M_{ix}(\gamma, \hat{\gamma}) = \sum_{l\in R_i}\left\{\frac{1}{\gamma}\frac{\bar{P}_{-i}^l}{N_l - 1} + \frac{1}{2\hat{\gamma}}\left(3\zeta_{+1}^l + \zeta_{+2}^l\right) + \frac{1}{\hat{\gamma}^2}\left| -p_{i2}^l + \frac{\bar{P}_{-i}^l}{N_l - 1} + \zeta_{+1}^{l2} + \zeta_{+1}^l\zeta_{+2}^l + \zeta_{+2}^{l2}\right|\right\} \quad (27)$$

$$M_{ip}^l(\gamma, \hat{\gamma}) = \frac{1}{\gamma}\frac{\bar{P}_{-i}^l}{N_l - 1} + \frac{1}{2\hat{\gamma}}\left(3\zeta_{+1}^l + \zeta_{+2}^l\right). \quad (28)$$

Coefficients M_{ix} and M_{ip}^l are the decreasing functions of γ and $\hat{\gamma}$. Since each $\mathcal{S}_i$ is compact, M_{ix} and M_{ip}^l have bounded maximums on $\mathcal{S}_F$. Hence, there exist sufficiently large γ and $\hat{\gamma}$ to satisfy the following inequalities in the entire space $\mathcal{S}_F$ so that the right-hand side of (26) is always nonnegative:

$$\alpha_i - 2M_{ix}(\gamma, \hat{\gamma}) \geq 0, \quad i \in \mathbf{I}$$
$$1 - M_{ip}^l(\gamma, \hat{\gamma}) \geq 0, \quad i \in \mathbf{I}\ \forall l \in R_i. \quad (29)$$

Proof of Proposition 4: To prove the monotonicity of function $F(s, \bar{X}, \bar{P})$ in its first argument, s, with fixed $\bar{X}$ and $\bar{P}$, we show that there exist γ and $\hat{\gamma}$ such that the following relation holds:

$$\langle \Delta_s F(s, \bar{X}, \bar{P}), \Delta s\rangle = \sum_{i=1}^N \left[\Delta_{s_i} F_i(s_i, \bar{X}_i, \bar{P}_i)\right]^T \Delta s_i > 0.$$

So, it is sufficient to show that the following inequality (30) holds:

$$\left[\Delta_{s_i} F_i(s_i, \bar{X}_i, \bar{P}_i)\right]^T \Delta s_i = \left[\Delta_{s_i} F_{ix}(s_i, \bar{X}_i, \bar{P}_i)\right]^T \Delta x_i + \sum_{l\in R_i}\left[\Delta_{s_i} F_{ip_i^l}\left(s_i, \bar{X}_i, \bar{P}_i\right)\right]^T \Delta p_i^l > 0 \quad \forall i \in \mathbf{I}. \quad (30)$$

Using (13)–(15) with fixed $\bar{X}_i$ and $\bar{P}_i$, and with steps similar to the proof of Proposition 2, we obtain the following inequality:

$$\left[\Delta_{s_i} F_i(s_i, \bar{X}_i, \bar{P}_i)\right]^T \Delta s_i \geq (\alpha_i - Q_{ix})(\Delta x_i)^2 + \sum_{l\in R_i}\frac{N_l}{N_l - 1}\left(2 - Q_{ip}^l\right)\left(\Delta p_i^l\right)^2 \quad (31)$$

where

$$Q_{ix}(\gamma, \hat{\gamma}) = \sum_{l\in R_i}\frac{N_l}{N_l - 1}Q^l(\gamma, \hat{\gamma}) \quad (32)$$

$$Q_{ip}^l(\gamma, \hat{\gamma}) = \frac{2}{\gamma}\cdot\frac{c_l - \bar{X}^l}{N_l} + Q^l(\gamma, \hat{\gamma})$$
$$Q^l = \left|\frac{1}{2N_l} + \frac{2}{\hat{\gamma}}\zeta_+^l + \frac{1}{\gamma}\frac{1}{N_l - 1}\left(\frac{N_l + 1}{N_l}\bar{P}^l - \left(p_{i1}^l + p_{i2}^l\right)\right)\right|. \quad (33)$$

Based on (31), it is sufficient that the following relations hold:

$$\alpha_i - Q_{ix}(\gamma, \hat{\gamma}) > 0 \quad \forall i \in \mathbf{I},$$
$$2 - Q_{ip}^l(\gamma, \hat{\gamma}) > 0 \quad \forall i \in \mathbf{I} \quad \forall l \in R_i. \quad (34)$$

Now, according to (32)–(34), by choosing γ and $\hat{\gamma}$ large enough, it suffices to have $2 - \frac{1}{2N_l} > 0, i \in \mathbf{I}\ \forall l \in R_i$ and $\alpha_i - \frac{1}{2}\sum_{l\in R_i}\frac{1}{N_l - 1} > 0, i \in \mathbf{I}$. The former is always true, and the latter defines the class of functions, which is defined in the statement of Proposition 4, and the proof is completed.

Proposition 6: Let Assumption 3 holds. Then, all operators $F_i(s_i, \bar{X}_i, \bar{P}_i)$ are uniformly Lipschitz continuous over $(\bar{X}_i, \bar{P}_i) \in \mathcal{X}_i \times \mathcal{P}_i$, i.e., for every fixed $s_i \in \mathcal{S}_i$, there is a positive constant $\bar{L}_i$ such that the following inequality holds:

$$\|\Delta_{\bar{X}_i, \bar{P}_i} F_i(s_i, \bar{X}_i, \bar{P}_i)\| \leq \bar{L}_i\sqrt{\|\Delta\bar{X}_i\|^2 + \|\Delta\bar{P}_i\|^2}. \quad (35)$$

Proof:

$$\|\Delta_{\bar{X}_i, \bar{P}_i} F_i(s_i, \bar{X}_i, \bar{P}_i)\|^2 = \|\Delta_{\bar{X}_i, \bar{P}_i} F_{ix}\left(s_i, \bar{X}_i, \bar{P}_i\right)\|^2 + \sum_{l\in R_i}\|\Delta_{\bar{X}_i, \bar{P}_i} F_{ip_i^l}\left(s_i, \bar{X}_i, \bar{P}_i\right)\|^2. \quad (36)$$

Substituting F_{ix} and $F_{ip_i^l}$ from (14) to (15), and using the facts that $(y_1 + y_2 + \cdots + y_q)^2 \leq q(y_1^2 + y_2^2 + \cdots + y_q^2)$ as well as $\Delta\zeta_+^{l2} \leq \Delta\bar{X}^{l2}/\hat{\gamma}^2$, $\Delta(\zeta_+^l\bar{P}^l) = \zeta_{+2}^l\Delta\bar{P}^l + \bar{P}_1^l\Delta\zeta_+^l$, $\Delta(\bar{P}^l\bar{X}^l) = \bar{X}_2^l\Delta\bar{P}^l +$

$\bar{P}_1^l \Delta \bar{X}^l$, and after some algebraic manipulation, the following inequalities are resulted:

$$\|\Delta_{\bar{X}_i,\bar{P}_i} F_{ix}(s_i,\bar{X}_i,\bar{P}_i)\|^2 \leq \sum_{l\in R_i} 2q\left\{a_{i\zeta}^{l2}(\Delta\zeta_+^l)^2 + a_{ip}^{l2}(\Delta\bar{P}^l)^2\right\}$$

$$\leq \sum_{l\in R_i} 2q\left\{\frac{a_{i\zeta}^{l2}}{\hat{\gamma}^2}(\Delta\bar{X}^l)^2 + a_{ip}^{l2}(\Delta\bar{P}^l)^2\right\}$$

$$\|\Delta_{\bar{X}_i,\bar{P}_i} F_{ip_i^l}(s_i,\bar{X}_i,\bar{P}_i)\|^2 \leq 3b_{ix}^{l2}(\Delta\bar{X}^l)^2 + 3b_{i\zeta}^{l2}(\Delta\zeta_+^l)^2$$

$$+ 3b_{ip}^{l2}(\Delta\bar{P}^l)^2 \leq 3\left(b_{ix}^{l2} + \frac{b_{i\zeta}^{l2}}{\hat{\gamma}^2}\right)(\Delta\bar{X}^l)^2 + 3b_{ip}^{l2}(\Delta\bar{P}^l)^2$$

where

$$a_{i\zeta}^l = \frac{4}{\hat{\gamma}}\left\{\frac{-N_l}{N_l-1}\left(p_i^l - \frac{\bar{P}_1^l}{N_l}\right) + \zeta_{+1}^{l2} + \zeta_{+1}\zeta_{+2} + \zeta_{+2}^{l2}\right\}$$

$$a_{ip}^l = \frac{1}{N_l-1}\left\{1 + \frac{4}{\hat{\gamma}}\zeta_{+2}^l + \frac{2}{\gamma}\cdot\frac{\bar{P}_1^l + \bar{P}_1^2 - (N_l+1)p_i^l}{N_l-1}\right\}$$

$$b_{ix}^l = -\frac{2}{\gamma}\frac{1}{N_l-1}\left(-p_i^l + \bar{P}_1^l\right),\ \ b_{i\zeta}^l = -2(\zeta_{+1}^l + \zeta_{+2}^l)$$

$$b_{ip}^l = -\frac{2}{N_l-1}\left(1 + \frac{\bar{X}_2^l - c_l}{\gamma}\right).$$

Thus, for the left-hand side of (36), we can write

$$\|\Delta_{\bar{X}_i,\bar{P}_i} F_i(s_i,\bar{X}_i,\bar{P}_i)\|^2 \leq \sum_{l\in R_i}\left\{A_i^{l2}\left(\Delta\bar{X}^l\right)^2 + B_i^{l2}\left(\Delta\bar{P}^l\right)^2\right\}$$

$$\leq \bar{L}_i^2 \sum_{l\in R_i}\left[\left(\Delta\bar{X}^l\right)^2 + (\Delta\bar{P}^l)^2\right].$$

Here, $A_i^{l2} = (2qa_{i\zeta}^{l2} + b_{i\zeta}^{l2})/\hat{\gamma}^2 + 3b_{ix}^{l2}$, $B_i^{l2} = 2qa_{ip}^{l2} + 3b_{ip}^{l2}$. Since all sets are convex and compact, all coefficients have maximum values inside their domains, and $\bar{L}_i = \sup_{l\in R_i, s\in\mathcal{S}_F}\{|A_i^l|, |B_i^l|\}$. By $s\in\mathcal{S}_F$, we mean $s_i\in\mathcal{S}_i$, $\bar{X}_i\in\bar{\mathcal{X}}_i$, and $\bar{P}_i\in\bar{\mathcal{P}}_i$ for all $i\in\mathbf{I}$.

Lemma 2 ([21], Lemma 2): Assume $W^l(k)$ is based on Assumption 6. In each iteration k, the following relations hold:

$$\sum_{i\in g^l}\tilde{X}_i^l(k) = \sum_{i\in g^l} x_i(k) = \bar{X}^l,\ \sum_{i\in g^l}\tilde{P}_i^l(k) = \sum_{i\in g^l} p_i^l(k) = \bar{P}^l.$$

Lemma 3: Assume that Assumption 6 holds and γ and $\hat{\gamma}$ are chosen to make Propositions 2 and 4 true. Then, there is a positive constant C that satisfies the following inequalities:

$$\begin{aligned}&\|F_i(s_i(k), \textstyle\sum\tilde{X}_i(k), \sum\tilde{P}_i(k))\| \leq C\\ &\|F_i(s_i(k), \hat{X}_i(k), \hat{P}_i(k))\| \leq C\end{aligned} \tag{37}$$

where

$$\sum\tilde{X}_i(k) = \left(\sum_{j\in g^{l_1^i}}\tilde{X}_j^{l_1^i}(k)\ \cdots\ \sum_{j\in g^{l_{q_i}^i}}\tilde{X}_j^{l_{q_i}^i}(k)\right)^T$$

$$\hat{X}_i(k) = \left(N_{l_1^i}\tilde{X}_i^{l_1^i}(k)\ \cdots\ N_{l_{q_i}^i}\tilde{X}_i^{l_{q_i}^i}(k)\right)^T$$

$$\sum\tilde{P}_i(k) = \left(\sum_{j\in g^{l_1^i}}\tilde{P}_i^{l_1^i}(k)\ \cdots\ \sum_{j\in g^{l_{q_i}^i}}\tilde{P}_j^{l_{q_i}^i}(k)\right)^T$$

$$\hat{P}_i(k) = \left(N_{l_1^i}\tilde{P}_i^{l_1^i}(k)\ \cdots\ N_{l_{q_i}^i}\tilde{P}_i^{l_{q_i}^i}(k)\right)^T.$$

Proof: Since Assumption 6 holds, so does Lemma 2. Thus, the first inequality could be rewritten as follows:

$$\|F_i(s_i(k), \textstyle\sum\tilde{X}_i(k), \sum\tilde{P}_i(k))\| = \|F_i(s_i(k), \bar{X}_i(k), \bar{P}_i(k))\|.$$

Since each $\mathcal{S}_i$ is compact, so are $\bar{\mathcal{X}}_i$ and $\bar{\mathcal{P}}_i$. Also, F_i is continuous on compact set $\mathcal{S}_i\times\bar{\mathcal{X}}_i\times\bar{\mathcal{P}}_i$. Hence, it is bounded and the first inequality in (37) holds.

Since each F_i is Lipschitz (see Proposition 6), we have

$$\|F_i(s_i(k), \hat{X}_i(k), \hat{P}_i(k))\| \leq \|F_i(s_i(k), \bar{X}_i(k), \bar{P}_i(k))\|$$

$$+ \|F_i(s_i(k), \hat{X}_i(k), \hat{P}_i(k)) - F_i(s_i(k), \bar{X}_i(k), \bar{P}_i(k))\|$$

$$\leq \|F_i(s_i(k), \bar{X}_i(k), \bar{P}_i(k))\|$$

$$+ \bar{L}_i\sqrt{\|\hat{X}_i(k) - \bar{X}_i(k)\|^2 + \|\hat{P}_i(k) - \bar{P}_i(k)\|^2}\,.$$

Also, $\hat{X}_i^l(k)\in\hat{\mathcal{X}}_i^l$ and $\hat{P}_i^l(k)\in\hat{\mathcal{P}}_i^l$, where $\hat{\mathcal{X}}_i^l$ and $\hat{\mathcal{P}}_i^l$ are the convex hull of strategy spaces of the users competing on link l. Definitely, $\bar{X}^l(k)\in\hat{\mathcal{X}}_i^l$ and $\bar{P}^l(k)\in\hat{\mathcal{P}}_i^l$ too. Similarly, $\hat{\mathcal{X}}_i^l$, $\hat{\mathcal{P}}_i^l\forall i\in\mathbf{I}, l\in R_i$ are compact due to the compactness of $\mathcal{S}_i\forall i\in\mathbf{I}$. Thus, $\|\hat{X}_i(k) - \bar{X}_i(k)\|$ as well as $\|\hat{P}_i(k) - \bar{P}_i(k)\|$ are bounded, and the second inequality is proved.

Proof of Proposition 5: The proof for the case of one connectivity graph is presented in [21]. Here, we aim to extend the proof to the case of multiple connectivity graphs. So far, we have proved all parts that depend on multiple graphs [see Propositions (2), (4), and (6), and Lemmas (2) and (3)]. For the rest of proof, which do not depend on our setting, readers are referred to [21].

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