Multipopulation Aggregative Games: Equilibrium Seeking via Mean-Field Control and Consensus

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***Abstract*—In this article, we extend the theory of deterministic mean-field/aggregative games to multipopulation games. We consider a set of populations, each managed by a population coordinator (PC), of selfish agents playing a global noncooperative game, whose cost functions are affected by an aggregate term across all agents from all populations. In particular, we impose that the agents cannot exchange information between themselves directly; instead, only a PC can gather information on its own population and exchange local aggregate information with the neighboring PCs. To seek an equilibrium of the resulting (partial-information) game, we propose an iterative algorithm where each PC broadcasts a mean-field signal, namely, an estimate of the overall aggregative term, to its own population only. In turn, we let the local agents react with the best response and the PCs cooperate for estimating the aggregative term. Our main technical contributions are to cast the proposed scheme as a fixed-point iteration with errors, namely, the interconnection of a Krasnoselskij–Mann iteration and a linear consensus protocol, and, under a nonexpansiveness condition, to show convergence towards an ε-Nash equilibrium, where ε is inversely proportional to the population size.**

***Index Terms*—Aggregative games, consensus, mean-filed, multipopulation, network games.**

I. Introduction

Distributed optimization and computational game theory for large-scale multiagent systems have been strong recent research areas in systems and control [1] with a variety of applications, e.g., in wireless communication [2], transmission networks [3], social networks [4], and smart grids [5].

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For multiagent systems with competitive agents, noncooperative game theory offers the mathematical background for equilibrium analysis [6]. Due to their potential to amend computation complexity and communication requirements, in this article, we are interested in the class of *aggregative* games [7], [8], where each agent is not subject to one-to-one dependence on nor interaction with other agents, but instead is subject to an aggregate effect from the whole population.

The literature on equilibrium seeking in aggregative games can be classified into two main approach classes in terms of information structure. In the first main class (full-information case), the agents have no information about the strategies of the other agents. Thus, they have to communicate with a population coordinator (PC), who can measure the aggregate term among all agents and broadcast a common signal to them [9]–[11], see [12] for a recent survey—this signal, commonly known as mean-field term (MFT), typically represents an estimation of the aggregate term. We refer to [13] for a recent application to energy management. In the second class of aggregative games (partial-information case), there is no population coordinator. Instead, the agents exchange information with each other through a network [14], where the aggregative term can be defined either as local [15] or global [16]. Recently, the authors in [17] proposed a fully distributed algorithm for seeking a generalized Nash equilibrium that exploits an interconnection of dynamic tracking of the aggregate term, projected-pseudo-gradient dynamics, and Krasnoselskij–Mann (KM) iterations.

In this article, we investigate a conceptually novel, hierarchical, approach amidst the two classes mentioned above. Our motivation is that reaching all agents with broadcast signals may be impossible in large-scale systems, especially in geographically distributed or heterogeneously networked systems, e.g., PEV parking lots, wireless networks [2], transportation networks [18]. On the other hand, a peer-to-peer communication network between selfish agents is typically not available or affordable, especially in engineering applications [19]–[21]. Our idea is then to cluster a large population of agents into multiple populations. In this way, the individual agents should not exchange information among each other, but only with their local population coordinator (PC). At the higher level, the PCs should exchange information in order to estimate the overall mean-field term (MFT). We believe this approach is a practical way to decouple selfish computations from the necessary information exchange. In our framework, the game is defined only among the agents of all populations, and therefore, the coordinators are not competitors nor payoff maximizers. Their role is only to estimate the overall aggregate strategy and inform the agents about it. This is to facilitate the information exchange and to preserve the personal information of each individual agent from its competitors.

Let us introduce our setup briefly. At the top level, we consider that each PC can exchange local estimates of the MFT with its neighboring PCs through a directed communication graph. At the bottom level, the members of each population are selfish agents who aim at minimizing their cost function without having information on the strategies of the others. However, the cost function of each agent is coupled with all other agents through the common MFT, which is a weighted average of the strategies of all agents from all populations. This motivates us to let each local agent apply a best response to the local MFT estimate only. Since the true MFT is unknown to the agents, its equilibrium value should be in turn estimated by the PCs.

To the best of our knowledge, we are the first to address multipopulation aggregative games via the interconnection of semidecentralized mean-field control and consensus theory. The main technical contributions of the article are as follows.

1) We extend the theory of deterministic mean-field/aggregative games [10], [11] to multipopulation games where the coordinators exchange information through a directed network. In our model, we have two different classes of agents, the population coordinators (estimators) and the population agents (decision makers), with coupled tasks.
2) In this coupled estimation and decision making setup, we propose an equilibrium seeking algorithm, a KM iteration with errors interconnected with a consensus protocol, with proven convergence to the mean-field term.
3) We prove that our algorithm converges to an ε-Nash equilibrium of the game, where ε is inversely proportional to the population size.

The article is structured as follows. The problem formulation of multipopulation aggregative game is introduced in Section II. Section III proposes an optimization-and-consensus method for the multipopulation aggregative game. The convergence analysis and the equilibrium of the game are discussed in Section IV. Finally, the proposed method is simulated on the plug-in electric vehicles' charging problem in Section V.

Notation: $\mathbb{N}$ and $\mathbb{R}$ are the set of natural and real numbers, respectively. $x^\top$ and $||x||$ denote the transpose and the Euclidean norm of the vector x, respectively. $\mathrm{col}(x_1,\ldots,x_N) = [x_1^\top,\ldots,x_N^\top]^\top$ and $x_{\mathcal{N}} = \mathrm{col}((x_n)_{n\in\mathcal{N}})$.

II. Multipopulation Aggregative Game Setup

We consider a set of P agent populations, each with a PC $p \in \mathcal{P} := \{1, 2, \ldots, P\}$, and with I_p agents, indexed by $\mathcal{I}_p$. The total number of agents is then $I := \sum_{p\in\mathcal{P}} I_p$.

We assume that the PCs can exchange information through a communication network defined by a possibly time-varying ($k \in \mathbb{N}$ being the discrete time index) directed graph $G(\mathcal{P}, \mathcal{E}^k)$, where $\mathcal{E}^k$ is a set of directed edges of graph G at time k of the decision making process. We consider that self loops are present, i.e., $(p,p) \in \mathcal{E}^k$, for $\forall p \in \mathcal{P}$. Let us denote the set of neighbors of PC p, namely, the set of PCs from which PC p can receive information, by $\mathcal{P}_p^k = \{p' \in \mathcal{P} \mid (p,p') \in \mathcal{E}^k\}$.

At the bottom level, each agent $i \in \mathcal{I}_p$ controls its decision $u_{p,i}$, which takes values in a compact and convex set $\mathcal{U}_{p,i} \subset \mathbb{R}^n$. To denote the collective decisions, we use the compact notations $\boldsymbol{u}_p := \mathrm{col}(u_{p,1},\ldots,u_{p,I_p})$ and $\boldsymbol{u} := \mathrm{col}(\boldsymbol{u}_1,\ldots,\boldsymbol{u}_P)$. Now, the aim of each agent i is to minimize its cost function

$$J_{p,i}(u_{p,i}, \sigma(\boldsymbol{u})) \tag{1}$$

where $\sigma(\boldsymbol{u})$ is an aggregative term, which we define as

$$\sigma(\boldsymbol{u}) = \frac{1}{P}\sum_{p\in\mathcal{P}}\sum_{i\in\mathcal{I}_p} \delta_{p,i}\, u_{p,i} \tag{2}$$

where $\delta_{p,i}$ are nonnegative coefficients such that $\sum_{i\in\mathcal{I}_p}\delta_{p,i} = 1$. Since the decisions of all agents affects the objective function in (1) of each agent through the function σ, we have a game equilibrium problem. We recall that a set of strategies in which no agent can benefit from individually deviating from its strategy is called Nash equilibrium. In our aggregative case, we adopt an analogous equilibrium concept.

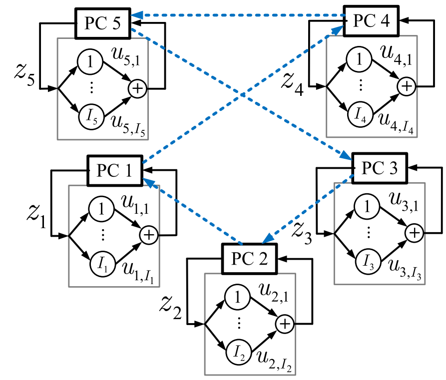

Fig. 1. Concept scheme of a multipopulation network.

Definition 1 (Multipopulation ε-Nash equilibrium): A set of strategies $\boldsymbol{u}^* \in (\mathbb{R}^n)^{PI}$ is a multipopulation ε-Nash equilibrium, with $\varepsilon > 0$, if, for all $p \in \mathcal{P}$ and $i \in \mathcal{I}_p$

$$J_{p,i}(u^*_{p,i}, \sigma(u^*)) \le \varepsilon + \min_{y\in\mathcal{U}_{p,i}} J_{p,i}\Bigg(y\,,\ \tfrac{1}{P}\delta_{p,i}\,y + \tfrac{1}{P}\sum_{i'\in\mathcal{I}_p\setminus\{i\}}\delta_{p,i'}\,u^*_{p,i'} + \tfrac{1}{P}\sum_{p'\in\mathcal{P}\setminus\{p\}}\sum_{i'\in\mathcal{I}_{p'}}\delta_{p',i'}\,u^*_{p',i'}\Bigg). \tag{3}$$

$\boldsymbol{u}^*$ is a multipopulation Nash equilibrium if (3) holds with $\varepsilon = 0$. □

III. Hierarchical Equilibrium Seeking: Mean-Field Control and Consensus

A. Local Best Responses and Global Information Exchange

We assume that the weights $\delta_{p,i}$ in (1) are determined by the PC p and the agents have no information about the function σ in (1), nor about the strategies of the other agents. Instead, we assume that each agent $i \in \mathcal{I}_p$ can only respond to a macroscopic signal, say $z_p \in \mathbb{R}^n$, here called local MFT (LMFT), which is broadcast by the PC p. The role of the signal z_p is to provide a local estimate of the global aggregative quantity $\sigma(\boldsymbol{u})$. Specifically, we assume that each agent i reacts optimally to the LMFT of its population via its best response

$$u^{\mathrm{br}}_{p,i}(\cdot) := \arg\min_{v\in\mathcal{U}_{p,i}} J_{p,i}(v,\cdot). \tag{4}$$

Let us group together the best response mappings into the following aggregative mappings

$$\forall p\in\mathcal{P}\colon\ \Lambda_p(z) := \sum_{i\in\mathcal{I}_p}\delta_{p,i}\,u^{\mathrm{br}}_{p,i}(z),\ \Lambda(z) := \frac{1}{P}\sum_{p\in\mathcal{P}}\Lambda_p(z). \tag{5}$$

The main idea here is that, for a large population size, all the LMFTs should converge to a fixed point of the mapping Λ in (5), in order to reach a multipopulation ε-Nash equilibrium. To this end, we set-up the following iterative procedure see Fig. 1 for a schematic representation. At every iteration k, each PC p broadcasts a signal z_p^k to its population. Then, each agent $i \in \mathcal{I}_p$ responds optimally according to (4), i.e., $u^k_{p,i} = u^{\mathrm{br}}_{p,i}(z^k_p)$. In turn, each PC p computes the aggregate quantity Λ^k_p over its population as

$$\Lambda^k_p = \Lambda_p(z^k_p). \tag{6}$$

Algorithm 1 Multi-population ε-Nash Equilibrium Seeking.

Initialization: $k \leftarrow 1,\ z_p^1 \leftarrow z_p^0\ ,\ \forall p \in \mathcal{P}$.
Iteration k:
Optimization: for each $p \in \mathcal{P}, i \in \mathcal{I}_p$:
$$u_{p,i}^k \leftarrow \underset{u_{p,i} \in \mathcal{U}_{p,i}}{\arg\min}\ J_{p,i}(u_{p,i}, z_p^k)$$
Aggregation: for each $p \in \mathcal{P}$:
$$\Lambda_p^k \leftarrow \sum_{i \in \mathcal{I}_p} \delta_{p,i} u_{p,i}^k$$
Communication and Update: for each $p \in \mathcal{P}$:
$$z_p^{k+1} \leftarrow (1-\alpha^k)\left(\sum_{p' \in \mathcal{P}} w_{p,p'}^k z_{p'}^k\right) + \alpha^k \Lambda_p^k$$
$k \leftarrow k+1$

It also receives $z_{p'}^k$ from each of its neighbor PCs, $p' \in \mathcal{P}_p^k$, through the communication network. Based on the local aggregation and the network information, each PC p updates its z_p^{k+1}, the LMFT estimate, to be broadcast at the next iteration.

B. Krasnoselskij–Mann Iteration and Consensus Protocol

In this section, we design how the PCs should update their LMFT estimates such that the iterative procedure outlined above converges and the resulting agent strategies reach a multipopulation ε-Nash equilibrium, where ε vanishes in the limit of infinite population size. Specifically, we exploit a KM fixed-point iteration and a consensus protocol on the LMFT estimates, simultaneously ($k \in \mathbb{N}$)

$$z_p^{k+1} = (1-\alpha^k)\left(\sum_{p' \in \mathcal{P}} w_{p,p'}^k\, z_{p'}^k\right) + \alpha^k \Lambda_p^k \tag{7}$$

where $\alpha^k \in (0,1), \forall k \geq 0$, are step sizes and $w_{p,p'}^k$ are communication weights. Precisely, let $W^k \in \mathbb{R}^{P \times P}$ be the weighted adjacency matrix of the communication graph G^k, where each element $w_{p,p'}^k$ of W^k satisfies $0 < w_{p,p'}^k \leq 1$ if $(p,p') \in \mathcal{E}^k$; $w_{p,p'} = 0$ implies no communication from p to p', i.e., $w_{p,p'}^k = 0$ for $(p,p') \notin \mathcal{E}^k$ at iteration k. Note that, we have $w_{p,p}^k > 0$ for each agent $\forall p \in \mathcal{P}$ and $\forall k \geq 0$. w_p^k indicates row p of matrix W^k. We are now ready to present our proposed equilibrium seeking algorithm, summarized in Algorithm 1, which is based on the two steps.

1) All agents compute in parallel their best response (4) to the local signal z_p; each PC p collects the aggregate Λ_p (5) among these best responses;
2) PC p updates z_p via (7) and broadcasts it to its agents.

IV. Convergence Analysis

We study the convergence of Algorithm 1 in three main steps. First, we show that when the PCs update their LMFTs as in (7), they reach to consensus on the MFT. Second, we show that the LMFTs converge to a fixed point of the mapping Λ in (5). Consequently, we prove that Algorithm 1 converges to a multipopulation ε-Nash equilibrium of the game.

A. Consensus of the Local Mean-Field Terms

In the following, we prove that all the LMFTs, $z_p^k\ \forall p \in \mathcal{P}$, converge to the same MFT, $\bar{z}^k$, over the whole population:

$$\bar{z}^k = \frac{1}{P} \textstyle\sum_{p \in \mathcal{P}} z_p^k. \tag{8}$$

With this aim, let us assume that the step sizes are slowly vanishing, that the communication graph is doubly stochastic, and that the union of the time-varying communication graphs is strongly connected over a finite horizon.

Assumption 1: The sequence $(\alpha^k)_{k \in \mathbb{N}}$ is nonincreasing, nonsummable, i.e., $\sum_k^\infty \alpha^k = \infty$, and square-summable, i.e., $\sum_k^\infty (\alpha^k)^2 < \infty$. □

Assumption 2: For each $k \in \mathbb{N}$, the adjacency matrix W^k is doubly stochastic, i.e.,

1) $w_{p,p'}^k \geq 0$, for all $k \geq 0$;
2) $\sum_{p' \in \mathcal{P}} w_{p,p'}^k = 1$, for all $p \in \mathcal{P}$, $k \geq 0$;
3) $\sum_{p' \in \mathcal{P}} w_{p',p}^k = 1$, for all $p \in \mathcal{P}$, $k \geq 0$.
4) $\exists \eta \in (0,1) : w_{p,p'}^k \geq \eta$, for all $(p,p') \in \mathcal{E}^k$, $k \geq 0$.

□

Assumption 3: There exists an integer $K \geq 1$ such that the graph $(\mathcal{P}, \bigcup_{k'=1}^K \mathcal{E}^{k'+k})$ is connected for all $k \geq 0$. □

We note that Assumption 3 ensures that each PC can indirectly reach all other PCs within K iterations, thus the information can spread throughout the entire network.

First, we characterize the evolution of the average among the LMFTs.

Lemma 1: $\bar{z}^k$ satisfies the following dynamics:

$$\bar{z}^{k+1} = (1-\alpha^k)\bar{z}^k + \frac{1}{P}\alpha^k \textstyle\sum_{p \in \mathcal{P}} \Lambda_p^k \tag{9}$$

with Λ_p as in (5). □

Proof: By averaging (7) over $\mathcal{P}$, we obtain

$$\begin{aligned}\bar{z}^{k+1} &:= \frac{1}{P}\sum_{p \in \mathcal{P}} z_p^{k+1} \\ &= \frac{1}{P}(1-\alpha^k)\sum_{p \in \mathcal{P}}\sum_{p' \in \mathcal{P}} w_{p,p'}^k z_{p'}^k + \frac{1}{P}\alpha^k \textstyle\sum_{p \in \mathcal{P}} \Lambda_p^k \end{aligned} \tag{10}$$

For the first term in (10), considering Assumption 2, we have

$$\sum_{p \in \mathcal{P}}\sum_{p' \in \mathcal{P}} w_{p,p'}^k z_{p'}^k = \sum_{p' \in \mathcal{P}} z_{p'}^k \sum_{p \in \mathcal{P}} w_{p,p'}^k = P\bar{z}^k.$$

The proof follows by putting the latter back into (10). ■

We are now ready to prove the main result of this section, the consensus on the LMFTs among PCs.

Theorem 1: Let Assumptions 1–3 hold. Then, the sequences $((z_p^k)_{k \in \mathbb{N}})_{p \in \mathcal{P}}$ generated by Algorithm 1 reach consensus, i.e., $\lim_{k \to \infty} \max_{p \in \mathcal{P}} \|z_p^k - \bar{z}^k\| = 0$, with $\bar{z}^k$ as in (8). □

Proof: Let us introduce the transition matrix from time k' to $k > k' \geq 0$ as $\Phi^{k,k'} := W^k W^{k-1} \cdots W^{k'}$ with $\Phi^{k,k} := W^k$, for all k, and let $[\Phi^{k,k'}]_{s,j}$ denote its (s,j) element. By (7), we can relate the LMFT of each PC p from time k' to $k+1$ as follows:

$$z_p^{k+1} = \alpha^k \Lambda_p^k + \sum_{r=k'}^{k-1}\sum_{j \in \mathcal{P}} [\bar{\Phi}^{k,r+1}]_{p,j}\alpha^r \Lambda_j^r + \sum_{j \in \mathcal{P}} [\bar{\Phi}^{k,k'}]_{p,j} z_p^{k'} \tag{11}$$

where $\bar{\Phi}^{k,k'} = \Phi^{k,k'} T^{k,k'}$, and $T^{k,k'} = (1-\alpha^k)(1-\alpha^{k-1})\cdots(1-\alpha^{k'})$. Then by using (9) from k' to k, we get

$$\bar{z}^{k+1} = T^{k,k'}\bar{z}^{k'} + \frac{1}{P}\sum_{r=k'}^{k-1}\sum_{p \in \mathcal{P}} T^{k,r+1}\alpha^r \Lambda_p^r + \frac{1}{P}\alpha^k \sum_{p \in \mathcal{P}} \Lambda_p^k. \tag{12}$$

Next, by substituting $k' \leftarrow 0$ and $k \leftarrow k-1$ in (11) and (12), and by computing the distance of a LMFT to the MFT, we derive the following inequality:

$$||z_p^k - \bar{z}^k|| \leq \sum_{j \in \mathcal{P}} \left| [\Phi^{k-1,0}]_{p,j} - \frac{1}{P} \right| T^{k-1,0} \, \|z_j^0\| + \sum_{r=0}^{k-2} \sum_{j \in \mathcal{P}} \left| [\Phi^{k-1,r+1}]_{p,j} - \frac{1}{P} \right| T^{k-1,r+1} \alpha^r \, \|\Lambda_j^r\| + \alpha^{k-1} \|\Lambda_p^{k-1}\| + \frac{1}{P} \alpha^{k-1} \sum_{j \in \mathcal{P}} \|\Lambda_j^{k-1}\|. \tag{13}$$

In view of Assumption 2 and 3, and [22, Prop. 1 (a)], we have that $|[\Phi^{k,k'}]_{p,j} - \frac{1}{P}| \leq c(\beta)^{k-k'}$, for some $\beta \in (0,1)$ and $c > 0$. By compactness of $\mathcal{U}_{p,i}$, there exist a positive constant B such that $\|u_{p,i}^{\mathrm{br}}(\cdot)\| \leq B$, thus $\|\Lambda_p^k\| \leq B, \forall p \in \mathcal{P}$. Therefore, (13) can be rewritten as follows:

$$\|z_p^k - \bar{z}^k\| \leq c(\beta)^{k-1} \sum_{j=1}^{P} T^{k-1,k'} \, \|z_j^0\| + PcB \sum_{r=0}^{k-2} (\beta)^{k-r-2} T^{k-1,r+1} \alpha^r + 2\alpha^{k-1} B. \tag{14}$$

Since $||z_p^0|| \leq B_0$ and $T^{k-1,r+1} \leq 1$, we have

$$\|z_p^k - \bar{z}^k\| \leq c(\beta)^{k-1} P B_0 + 2\alpha^{k-1} B + PcB \sum_{r=0}^{k-2} (\beta)^{k-2-r} \alpha^r. \tag{15}$$

Finally, since $\lim_{k\to\infty} \alpha^k = 0$ and $0 < \beta < 1$, for $k \to \infty$, the first two addends in the right hand side of (15) converge to zero. Also, it holds that $\lim_{k\to\infty} \sum_{r=0}^{k} (\beta)^{k-r} \alpha^r = 0$. Consequently, $\lim_{k\to\infty} \|z_p^k - \bar{z}^k\| = 0$ for all $p \in \mathcal{P}$. ■

B. Convergence of the Mean-Field Term to a Fixed Point of the Aggregation Mapping

Our next step is to show that the consensus MFT, $\bar{z}^k$, converges to a fixed point of the aggregation mapping, Λ in (5). Before establishing our main result, we postulate a technical assumption and a preliminary lemma on the boundedness of the MFT estimation error weighted by the vanishing step sizes.

Assumption 4: The functions $\{J_{p,i}\}_{p\in\mathcal{P}, i\in\mathcal{I}_p}$ in (1) are uniformly $\ell-$Lipschitz continuous, for some $\ell > 0$ independent of P and I, and all the best-response mappings $\{u_{p,i}^{\mathrm{br}}\}_{p\in\mathcal{P}, i\in\mathcal{I}_p}$ in (4) are nonexpansive. □

Remark 1: The nonexpansiveness in Assumption 4 can be efficiently checked for quadratic cost functions, (see [10]). □

Lemma 2: Let Assumptions 1–3 hold. Then $\sup_{p\in\mathcal{P}} \sum_{k=1}^{\infty} \alpha^k \|z_p^k - \bar{z}^k\| < \infty$. □

Proof: By (15), we have that

$$\sum_{k=1}^{\infty} \alpha^k \, \|z_p^k - \bar{z}^k\| \leq cPB_0 \sum_{k=1}^{\infty} \alpha^k (\beta)^{k-1} + 2B \sum_{k=1}^{\infty} \alpha^k \alpha^{k-1} + PcB \sum_{k=2}^{\infty} \alpha^k \sum_{r=0}^{k-2} (\beta)^{k-2-r} \alpha^r.$$

Thanks to Assumption 1 and $\beta \in (0,1)$, the terms $\sum_{k=1}^{\infty} \alpha^k (\beta)^{k-1}$ and $\sum_{k=1}^{\infty} \alpha^k \alpha^{k-1}$ are bounded. For the last term, $\sum_{k=2}^{\infty} \alpha^k \sum_{r=0}^{k-2} (\beta)^{k-2-r} \alpha^r$, we can exploit the fact that α^k is nonincreasing

$$\sum_{k=2}^{\infty} \alpha^k \sum_{r=0}^{k-2} (\beta)^{k-2-r} \alpha^r \leq \sum_{k=2}^{\infty} \sum_{r=0}^{k-2} (\beta)^{k-2-r} (\alpha^r)^2 = \sum_{r=0}^{\infty} (\alpha^r)^2 \sum_{k=r}^{\infty} (\beta)^{k-r} = (\beta)^{-1} \sum_{r=0}^{\infty} (\alpha^r)^2 < \infty. \tag{16}$$

It then follows that $\sum_{k=1}^{\infty} \alpha^k \|z_p^k - \bar{z}^k\| < \infty$. ■

We are ready to show the convergence of all the LMFTs to a fixed point of the aggregation mapping in (5).

Assumption 5: The coefficients $(\delta_{p,i})_{p\in\mathcal{P}, i\in\mathcal{I}_p}$ in (2) are nonnegative, such that for all $p \in \mathcal{P}$, $\sum_{i\in\mathcal{I}_p} \delta_{p,i} = 1$, and are uniformly bounded, i.e., $\delta_{p,i} \leq c/I_p$, for some constant $c > 0$ independent of P and I. □

Theorem 2: Let Assumptions 1–5 hold. The collective sequence $((z_p^k)_{p\in\mathcal{P}})_{k\in\mathbb{N}}$ generated by Algorithm 1 converges to a fixed point of the aggregation mapping $\Lambda(\cdot)$ in (5). □

Proof: By (9) in Lemma 1, we have

$$\bar{z}^{k+1} = (1 - \alpha^k) \bar{z}^k + \alpha^k \Lambda(z_p^k). \tag{17}$$

Let define the aggregation error as $e^k = \Lambda(z_p^k) - \Lambda(\bar{z}^k)$. Therefore, (17) can be rewritten as

$$\bar{z}^{k+1} = (1 - \alpha^k) \bar{z}^k + \alpha^k (\Lambda(\bar{z}^k) + e^k). \tag{18}$$

By the definition of $\Lambda(\cdot)$ in (5) and Assumption 4, we note that $\Lambda(\cdot)$ is a convex combination of nonexpansive mappings, hence it is nonexpansive as well. Thus, according to [23, Th. 3.3], $\bar{z}^k$ in (18) converges to a fixed point of $\Lambda(\cdot)$ in (5) if $\sum_{k=1}^{\infty} \alpha^k (1 - \alpha^k) = \infty$ and $\sum_{k=1}^{\infty} \alpha^k ||e^k|| < \infty$ hold true. The first condition follows directly from Assumption 1. For the second condition, we can write

$$\sum_{k=1}^{\infty} \alpha^k ||e^k|| = \sum_{k=1}^{\infty} \alpha^k ||\Lambda(z_p^k) - \Lambda(\bar{z}^k)|| \leq \frac{1}{P} \sum_{p\in\mathcal{P}} \sum_{k=1}^{\infty} \alpha^k ||\Lambda_p(z_p^k) - \Lambda_p(\bar{z}^k)|| \leq \frac{1}{P} \sum_{p\in\mathcal{P}} \sum_{k=1}^{\infty} \alpha^k ||z_p^k - \bar{z}^k||. \tag{19}$$

Now by applying Lemma 2 to the last term, we get $\sum_{k=1}^{\infty} \alpha^k ||e^k|| < \infty$ as desired. Consequently, $\bar{z}^k$ converges to a fixed point of $\Lambda(\cdot)$. In view of Theorem 1, we conclude that each $z_p^k, \forall p \in \mathcal{P}$, converges to such a fixed point. ■

Remark 2: The range of the mapping Λ is bounded due to the compactness of the sets $\mathcal{U}_{p,i}$ and the uniform boundedness of the coefficients $\delta_{p,i}$ (Assumption 5). Therefore, according to [24, Th. 4.1.5 (b)], the mapping $\Lambda(\cdot)$ has a fixed point. □

C. Multipopulation ε-Nash Equilibrium Analysis

We conclude the technical part of the article by showing that the outcome of Algorithm 1, a set of strategies that are best responses to a fixed point of the aggregation mapping in (5) is a multipopulation ε-Nash equilibrium, where ε is inversely proportional to the population size.

Assumption 6: There exists a lower bound on the number of agents of all local populations which is uniformly proportional to the average population size, i.e., $\min_{p\in\mathcal{P}} I_p \geq d\frac{I}{P}$, for some constant $d \in (0,1)$ independent of P and I. □

Theorem 3: Let Assumptions 4, 5, and 6 hold and let $\bar{z} = \mathrm{col}(\bar{z}_1, \ldots, \bar{z}_P)$ be a fixed point of the aggregation mapping $\Lambda(\cdot)$ in

(5). Then, the associated collective set of best-response strategies, i.e., $\{\bar{u}_{p,i} := \text{argmin}_{v \in \mathcal{U}_{p,i}} J_{p,i}(v, \bar{z}_p) \mid p \in \mathcal{P},\, i \in \mathcal{I}_p\}$, is a multipopulation ε-Nash equilibrium, where ε is inversely proportional to I. □

Proof: Let us define:

$$u^\bullet_{p,i} := u^{\text{br}}_{p,i}(z^\bullet),\;\; \hat{u}_{p,i} := u^{\text{br}}_{p,i}(z'_{p,i}),$$
$$\tilde{u}_{p,i} := u^{\text{br}}_{p,i}\left(\frac{1}{P}\delta_{p,i}u_{p,i} + v^\bullet_{p,i}\right) \tag{20}$$

where $z^\bullet = \Lambda(z^\bullet) = \lim_{k\to\infty}\bar{z}^k$ is the converged MFT, $v^\bullet_{p,i} = z^\bullet - \frac{1}{P}\delta_{p,i}u^\bullet_{p,i}$ is obtained by excluding the effect of $u^\bullet_{p,i}$ from $z^\bullet$ and $z'_{p,i} = \frac{1}{P}\delta_{p,i}\tilde{u}_{p,i} + v^\bullet_{p,i}$. Then it follows that:

$$J_{p,i}(\hat{u}_{p,i}, z'_{p,i}) \le J_{p,i}(\tilde{u}_{p,i}, z'_{p,i}) \le J_{p,i}(u^\bullet_{p,i}, z^\bullet). \tag{21}$$

The first inequality holds because $\hat{u}_{p,i}$ is the best response to $z'_{p,i}$, while $\tilde{u}_{p,i}$ is not. The second inequality holds since the best response of an agent to the mean-field term results in a higher cost compared to the case in which the agent optimizes over the effect of its own strategy ($\frac{1}{P}\delta_{p,i}u_{p,i}$) in the mean-field term as well. Therefore, since $J_{p,i}$ is Lipschitz continuous according to Assumption 4, by (3) and (21), we conclude that

$$J_{p,i}(u^\bullet_{p,i}, z^\bullet) - J_{p,i}(\tilde{u}_{p,i}, z'_{p,i}) \le J_{p,i}(u^\bullet_{p,i}, z^\bullet) - J_{p,i}(\hat{u}_{p,i}, z'_{p,i})$$
$$\le \ell(||u^\bullet_{p,i} - \hat{u}_{p,i}|| + ||z^\bullet - z'_{p,i}||)$$

where ℓ is the Lipschitz constant of $J_{p,i}$. Then, by using Assumption 4 and the definition of $z^\bullet$ and $z'_{p,i}$, we have

$$J_{p,i}(u^\bullet_{p,i}, z^\bullet) - J_{p,i}(\tilde{u}_{p,i}, z'_{p,i})$$
$$\le 2\ell||z^\bullet - z'_{p,i}|| \le \frac{2}{P}\ell\delta_{p,i}||\tilde{u}_{p,i} - u^\bullet_{p,i}||.$$

Since $\mathcal{U}_{p,i}$ are compact, we have $||\tilde{u}_{p,i} - u^\bullet_{p,i}|| \le ||\tilde{u}_{p,i}|| + ||u^\bullet_{p,i}|| \le 2B$. Therefore, by Assumptions 5 and 6, we obtain

$$J_{p,i}(u^\bullet_{p,i}, z^\bullet) - J_{p,i}(\tilde{u}_{p,i}, z'_{p,i})$$
$$\le \frac{4}{P}B\ell\delta_{p,i} \le \frac{4B\ell c}{PI_p} \le \frac{4B\ell cd}{I} =: \varepsilon. \tag{22}$$

Thus, the solution generated by Algorithm 1 is an ε-Nash equilibrium with ε in (22), the constant $4B\ell cd$ does not depend on P and I.

Remark 3: In view of Remark 2, Theorems 2 and 3, under suitable assumptions, Algorithm 1 converges to a set of strategies that is an ε-Nash equilibrium, where in view of (22) in the proof of Theorem 3, ε is upper bounded by a uniform constant divided by the overall population size, I. Therefore, whenever all the local population sizes are very large (infinite), the ε is very small (zero). □

Remark 4: Based on (15) and the sublinear convergence of the KM iteration (18) [25, Th. 1], one can show that Algorithm 1 converges at least with R-sublinear order of convergence [26, Sec. 9.2]. □

V. Illustrative Example: Charging Plug-In Electric Vehicles in Multiple Parking Lots

We consider the charging coordination problem for a population of plug-in electric vehicles (PEVs) that have been distributed in P different parking lots. We consider that each parking lot p has a coordinator who can exchange information with some other parking lot coordinators through a communication graph $G(\mathcal{P}, \mathcal{E})$. We assume that each PEV $i \in \mathcal{I}_p$ is located in the parking $p \in \mathcal{P}$ in some time slots $\mathcal{T} = \{1, \dots, T\}$. Each PEV $i \in \mathcal{I}_p$ controls its charged energy $u_{p,i} = [u^1_{p,i}, \dots, u^T_{p,i}]^\top$,

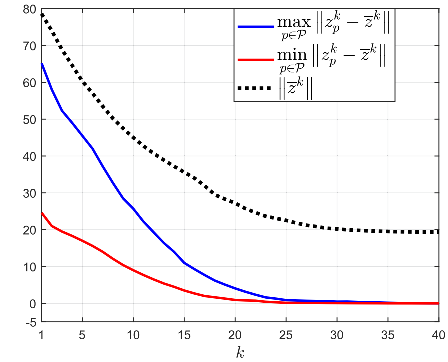

Fig. 2. Evolution of the LMFTs (solid lines) toward consensus and the MFT (dashed line) towards a fixed point of the aggregation mapping.

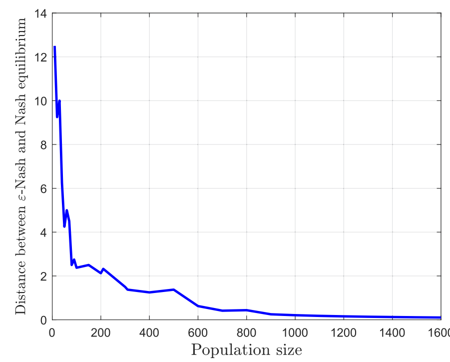

Fig. 3. Distance between the computed ε-Nash equilibrium and a Nash equilibrium versus the population size.

where $u^t_{p,i}$ indicates the energy demand at time slot $t \in \mathcal{T}$, which must satisfy the constraint $\mathcal{U}_{p,i} := \{u^t_{p,i} | \underline{u}_{p,i} \le u^t_{p,i} \le \overline{u}_{p,i}\}$. Also, let $x^t_{p,i}$ denotes the state of charge (SoC) of the battery of PEV $i \in \mathcal{I}_p$ at $t \in \mathcal{T}$, which we assume to evolve according to the dynamics

$$x^{t+1}_{p,i} = x^t_{p,i} + \frac{\eta_{p,i}}{\beta_{p,i}}u^t_{p,i}, u^t_{p,i} \in \mathcal{U}_{p,i} \tag{23}$$

where $\beta_{p,i}$ is the battery size, and $\eta_{p,i} \in (0, 1]$ is the charging efficiency. To ensure the appropriate functionality of the battery, the SoC must satisfy $\underline{x}_{p,i} \le x^t_{p,i} \le \overline{x}_{p,i}$, for all $t \in \mathcal{T}$. Each PEV $i \in \mathcal{I}_p$ aims to minimize its cost function

$$J_{p,i} = C_{p,i}(u_{p,i}) + \lambda(\boldsymbol{u})^\top u_{p,i} + \gamma_{p,i}(x^T_{p,i} - x^D_{p,i})^2 \tag{24}$$

where $C_{p,i}(u_{p,i}) = \sum_{t\in\mathcal{T}} q_{p,i}(u^t_{p,i})^2 + r_{p,i}u^t_{p,i} + h_{p,i}$ is a quadratic battery degradation cost as in [27] and $q_{p,i}, r_{p,i}, \gamma_{p,i}$, and $h_{p,i}$ are positive parameters. $x^D_{p,i}$ is the desired SoC for PEV i of parking p. $\lambda(\boldsymbol{u})$, $\boldsymbol{u} = \text{col}(\boldsymbol{u}_{\mathcal{I}_1}, \dots, \boldsymbol{u}_{\mathcal{I}_P})$ and $\boldsymbol{u}_{\mathcal{I}_p} = \text{col}(u_{p,1}, \dots, u_{p,I_p})$, indicates the price of energy which is defined as a congestion cost function [28]

$$\lambda(\boldsymbol{u}) = a\left(d + \sum_{p\in\mathcal{P}}\sum_{i\in\mathcal{I}_P} u_{p,i}\right) + b \tag{25}$$

where $a > 0$ and b is a positive vector, and the vector d represents the normalized demand from nonPEV electrical loads. We simulate the charging coordination problem over 24 time slots with $P = 10$ parking lots and the number of PEVs I_p for each parking lot is same; $I_1 = \cdots = I_P = 150$. We borrow the parameters of the cost functions from [28]. As for the electricity price in (25), we set $a = 1.1 \times 10^{-2}$, $b = 0.06 \cdot \mathbb{1}_T$ and the vector d empirically derived from [19]. Note that based on (2), if we have P parking lots with each contains I_{eq} PEVs, then a is adjusted by the coefficient $1500/(PI_{eq})$. We set the parameters of the battery degradation cost as $q_{p,i} = 1.25$, $r_{p,i} = 0.11$, and $h_{p,i} = 0.2$, lower bound of the input to 0 (to prevent V2G) and upper bound to 5 (kW). For each PEV, battery capacity size and charging efficiency are randomly selected in the range of [25,30] and [0.5,1], respectively. Also, all the PEVs share the common minimum and maximum SoC, 0.1 and 0.9, respectively. Furthermore, we set the step sizes of Algorithm 1 to $\alpha^k = 1/(k+1)$. In Fig. 2, we show the minimum and maximum distances between the LMFTs and the MFT, which represents the estimation error of the price signal. As expected, the local estimates, $(z_p^k)_{p\in\mathcal{P}}$, reach consensus (Theorem 1) and converge towards the equilibrium, since $\bar{z}^k$ converges as well (Theorem 2). Fig. 3 shows the distance between the computed ε-Nash equilibrium and a Nash equilibrium for different population sizes. We observe that for large population sizes, the ε becomes small.

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