Leader–Follower Network Aggregative Game With Stochastic Agents' Communication and Activeness

Mohammad Shokri and Hamed Kebriaei, *Senior Member, IEEE*

***Abstract*—This article presents a leader–follower scheme for network aggregative games. The followers and leader are selfish cost minimizing agents. The cost function of each follower is affected by its own strategy, the strategy of leader and the aggregate strategy of its neighbors through a communication graph. Also, the leader's cost function depends on its own strategy and the aggregate strategy of all the followers. The leader infinitely often wakes up, receives the aggregate strategy of the followers, updates its decision value, and broadcasts it to all the followers. Then, the followers apply the updated strategy of the leader into their cost functions. The establishment of information exchange between each neighboring pair of followers, and also, decision updating by a follower at each iteration, that is to say the *activeness* of the follower in that iteration, are both considered to be drawn from two arbitrary distributions. Moreover, a distributed algorithm based on subgradient method is proposed for updating the strategies of leader and followers. The convergence of the proposed algorithm to the unique Stackelberg equilibrium point of the game is proven in both almost sure and mean square senses.**

***Index Terms*—Distributed algorithm, leader–follower, network aggregative game (NAG), stochastic network, subgradient method.**

I. Introduction

Distributed optimization over networks has attracted widespread attention of researchers in recent years [1]. As a typical framework, each agent in a network aims to minimize a social or an individual cost function while it communicates with some other agents through the network. In case that each agent is modeled as a selfish player who aims to minimize its own cost and also, the agent's cost is affected by decision variables of its neighbors through the network topology, the problem can be studied as a noncooperative network game [2]. If the effect of decision variables of rivals on the agent's cost function appears as an aggregative term (e.g., summation or weighted sum), the network game is known as network aggregative game (NAG) [3]. Many applications can be studied via this framework including, power system [4], opinion dynamics [5], communication system [6], provision of public goods [7], and criminal networks [8].

In some researches on NAGs, it is assumed that the cost function of each agent is affected by the aggregate strategy of all network agents (including neighbors and nonneighbors) [9]–[11]. In this case, the coupling term among the agents is the same for all of them, and hence, this scheme does not cover the situations that agents have different interdependencies or limited communication capabilities. In another type of NAGs (local NAG), the cost function of each agent is affected only by its neighbors. In [12], a local NAG is studied for agents with quadratic cost functions. Two distributed algorithms are proposed to steer the strategies of players to the Nash equilibrium point, while at each iteration, the agents communicate with their neighbors and update their strategies based on the best response method. Furthermore, in [13], a Nash seeking dynamics is utilized for agents with proximal quadratic cost functions whose best responses are in the form of proximal operator. In [14], the cost function of the agents is considered in a general form and a distributed algorithm has been developed in which, at each iteration, each agent communicates with all its neighbors and updates its strategy based on its best response function.

In the mentioned research on NAGs, all of the agents decide simultaneously, which leads to a single level Nash game. However, in many applications, there is a high-level agent (leader), who aims to optimize its own objective function that also depends on the strategies of other agents at a lower level (followers). Several research have investigated leader–follower games [15]–[17] and it has been extensively utilized in many engineering fields such as wireless sensor networks [18], supply chain management [19], and smart grid [20]. If the leader has complete information about the followers' cost functions, then the concept of Stackelberg equilibrium (SE) can be applied directly [16]. However, if the leader does not have *a priori* information about the cost function of the followers, then the leader needs to learn its optimal strategy by iterative methods. In [15], a leader-following problem is discussed in which the cost function of the leader is independent of the followers' strategies and only the followers respond to leaders' strategy. In [17], an iterative hierarchical mean-field game is studied including a leader and a large number of followers. The leader first announces its decision and then, followers respond by knowing the leader's decision.

In this article, we propose a leader–follower NAG. The leader has a different type of cost function from the followers which is affected by the aggregated strategy of all the followers. Additionally, the leader has a different type of communication and activeness from the followers. It is considered that the leader infinitely often: wakes up, receives the last aggregated strategy of followers, updates its strategy, broadcasts it to all the followers, and then goes to sleep. In the lower level, the followers receive the last decision value of the leader and play a NAG until the next decision update of the leader. The cost function of each follower is affected by aggregated strategies of its neighbors and also the strategy of the leader. We also consider stochastic communication and activeness of the agents in NAG. At each arbitrary iteration, based on a stochastic binary distribution, each follower may become active

Manuscript received July 14, 2019; revised November 7, 2019; accepted February 7, 2020. Date of publication February 13, 2020; date of current version December 3, 2020. This work was supported in part by the Institute for Research in Fundamental Sciences (IPM) under Grant CS 1398-4-256. Recommended by Associate Editor R. P. Malhame.
(Corresponding author: Hamed Kebriaei.)

Mohammad Shokri is with the School of Electrical and Computer Engineering, College of Engineering, University of Tehran, Tehran 13145-1384, Iran (e-mail: mo.shokri@ut.ac.ir).

Hamed Kebriaei is with the School of Electrical and Computer Engineering, College of Engineering, University of Tehran, Tehran 13145-1384, Iran, and also with the School of Computer Science, Institute for Research in Fundamental Sciences (IPM), Tehran 19395-5746, Iran (e-mail: kebriaei@ut.ac.ir).

Color versions of one or more of the figures in this article are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TAC.2020.2973807

to update its decision based on the projected subgradient method. Besides, at each iteration, an agent may receive the decision value of a neighboring agent based on another stochastic binary distribution. The corresponding stochastic binary variables of the two mentioned distributions can be dependent on each other, and further, there might be some constraints on those variables. Finally, a distributed algorithm is proposed in which the decision values of the leader and followers converge to the unique SE point of the game.

To the best of our knowledge, compared to single-level NAGs [12]–[14], this is the first article that proposes a leader–follower scheme for NAGs. Further, this is the first article that presents a general stochastic framework that simultaneously considers communication and activeness of the agents in NAGs in which the Gossip-based communication protocol [21] can be encountered as a special case of the proposed framework. From other aspects, compared to aggregative games that consider the average strategy of whole population as a common coupling term among the agents [9]–[11], in this article, the local aggregative term is studied in which only neighbors of each follower, as well as the leader, affect the follower's cost function. Compared to the literature of NAGs, those consider the local aggregative term, we have studied a general strongly convex cost function instead of quadratic one [12], [13]. In contrast with the papers, which have utilized the best response dynamics as the agents' decision update rule [12]–[14], we have used the projected subgradient method for optimization to cope with the limited computational capabilities of the agents. The main contributions of this article can be summarized as follows.

1) We propose a leader–follower framework for NAG.
2) We study stochastic communication and activeness of the agents in NAG.
3) A distributed algorithm based on projected subgradient method is proposed and its convergence to the unique SE point of the game is proven in both almost sure and mean square senses.

Notation and Preliminaries

$\mathbb{N}$ and $\mathbb{R}$ are the set of natural and real numbers, respectively. $|\mathcal{N}|$ denotes the number of members of the set $\mathcal{N}$. Let $A^\top$ and $||A||$ denote the transpose and the norm-2 of a vector/matrix A, respectively. The column augmentation of vectors x_n for $n = 1, \dots, N$ is defined as $\mathbf{col}(x_1, \dots, x_N) = [x_1^\top, \dots, x_N^\top]^\top$. $\vec{1}_n = [1, \dots, 1]^\top$ and $\vec{0}_n = [1, \dots, 1]^\top$, where $\vec{1}_n, \vec{0}_n \in \mathbb{R}^n$. The probability function and expected value are denoted by $\mathbf{P}\{.\}$ and $\mathbf{E}\{.\}$, respectively. Supposing the function $f(.) : \mathcal{X} \to \mathbb{R}$, $g(x')$ is called the subgradient of $f(.)$ at x' if $\forall x \in \mathcal{X} : f(x') + (x - x')^\top g(x') \le f(x)$. Also, the projection operator of $\mathcal{X}$ is defined by $\Pi_{\mathcal{X}}(x) = \arg\min_{y \in \mathcal{X}} ||y - x||^2$. $g(x)$ is strictly monotone if $(g(x_2, r) - g(x_1, r))^\top (x_2 - x_1) > 0$ for $\forall x_1, x_2 : x_1 \ne x_2$.

II. System Model

Consider a set of follower agents $\mathcal{N} = \{1, \dots, N\}$ and a leader involved in a noncooperative game. The followers are connected to each other via a communication network represented by a directed graph $G(\mathcal{N}, \mathcal{A})$, where $\mathcal{A} = [a_{nm}]_{n,m\in\mathcal{N}}$ is the adjacency matrix of G such that $a_{nm} = 1$ if there is a communication link from follower m to n, and $a_{nm} = 0$ otherwise (more details on communication network of the followers is given in Section III-A). Each follower $n \in \mathcal{N}$ has its decision variable (i.e., strategy) $x_n \in \mathcal{X}_n$ where $\mathcal{X}_n \subset \mathbb{R}^{M^F}$ is a nonempty, compact, and convex set. Furthermore, $y \in \mathcal{Y}$ is the strategy of leader which is selected from a compact and convex set denoted by $\mathcal{Y} \subset \mathbb{R}^{M^L}$. The cost function of follower n depends on the aggregated strategy of its neighbors whose set is denoted by $\mathcal{N}_n$, and also the strategy of the leader. Therefore, the cost function of follower n is defined as follows:

$$J_n^F(x_n, \sigma_n(x_{-n}), y) : x_n \in \mathcal{X}_n \tag{1}$$

where $\sigma_n(x_{-n}) = \sum_{m\in\mathcal{N}_n} w_{nm} x_m$, $x_{-n} = \mathbf{col}(x_1, \dots, x_{n-1}, x_{n+1}, \dots, x_N)$, and w_{nm} is information weight of follower n from strategy of follower m. The weights satisfy $\sum_{m\in\mathcal{N}-\{n\}} w_{nm} = 1$, $w_{nm} > 0$ if $a_{nm} = 1$, and $w_{nm} = 0$ if $a_{nm} = 0$. Hence, we can define the weight matrix of the graph G by $\mathcal{W} = [w_{nm}]_{n,m\in\mathcal{N}}$. The cost function of the leader is defined as

$$J^L(y, \sigma_0(x_{\mathcal{N}})) : y \in \mathcal{Y} \tag{2}$$

where $\sigma_0(x_{\mathcal{N}}) = \sum_{n\in\mathcal{N}} w_{0n} x_n$, $x_{\mathcal{N}} = \mathbf{col}(x_1, \dots, x_N)$, w_{0n} denotes the weight of communication link between follower n and the leader. we have $\sum_{n\in\mathcal{N}} w_{0n} = 1$, and $w_{0n} \ge 0$. We let $\vec{w}_0 = \mathbf{col}(w_{01}, \dots, w_{0N})$ denotes the leader weight vector.

Accordingly, the noncooperative game among the followers and leader is defined as follows:

$$\mathcal{G} = \begin{cases} \text{Players: followers } \mathcal{N} \text{ and the leader} \\ \text{Strategies:} \begin{cases} \text{Follower } n\text{:} & x_n \in \mathcal{X}_n \\ \text{Leader:} & y \in \mathcal{Y} \end{cases} \\ \text{Cost:} \begin{cases} \text{Follower } n\text{:} & J_n^F(x_n, \sigma_n(x_{-n}), y) \\ \text{Leader:} & J^L(y, \sigma_0(x_{\mathcal{N}})). \end{cases} \end{cases} \tag{3}$$

In this article, we define $d_n(x_n, \sigma_n(x_{-n}), y)$ and $d_0(y, x_{\mathcal{N}})$ as a subgradient of $J_n^F(x_n, \sigma_n(x_{-n}), y))$ with respect to x_n, and $J^L(y, \sigma_0(x_{\mathcal{N}}))$ with respect to y, respectively.

Assumption 1: $J_n^F(x_n, \sigma_n(x_{-n}), y)$ and $J^L(y, \sigma_0(x_{\mathcal{N}}))$ are subdifferentiable and strongly convex over $\mathcal{X}_n$ and $\mathcal{Y}$ with respect to x_n and y, respectively, i.e., there exist C_n and C_0 for $\forall n \in \mathcal{N}$ such that

$$\begin{aligned} &(d_n(x_n, \sigma, y) - d_n(x_n', \sigma, y))^\top (x_n - x_n') \ge C_n ||x_n - x_n'||^2 \\ &(d_0(y, \sigma) - d_0(y', \sigma))^\top (y - y') \ge C_0 ||y - y'||^2. \end{aligned} \tag{4}$$

Also, there exist Lipschitz constants L and L_0 such that $\forall n \in \mathcal{N}, \forall x_n \in \mathcal{X}_n, y \in \mathcal{Y}, \forall \sigma_1, \sigma_2 \in \{\sigma_n(x_{-n}) | x_{-n} \in \prod_{m\ne n} \mathcal{X}_m\}$, we have

$$\begin{aligned} &||d_n(x_n, \sigma_1, y_1) - d_n(x_n, \sigma_2, y_2)|| \le L||\sigma_1 - \sigma_2|| \\ &\quad + L||y_1 - y_2|| \\ &||d_0(y, \sigma_1) - d_0(y, \sigma_2)|| \le L_0 ||\sigma_1 - \sigma_2|| \end{aligned} \tag{5}$$

where $d_n(x_n, \sigma_n(x_{-n}), y)$ and $d_0(y, \sigma_n(x_{-n}))$ are the subgradients of J_n^F and J^L, respectively. □

The equilibrium point of the aforementioned leader–follower game is defined as follows:

Definition 1: $(x_{\mathcal{N}}^*, y^*)$ is a SE point of the leader–follower game among the followers and the leader if

$$\begin{aligned} &\forall x_n \in \mathcal{X}_n : J_n^F(x_n^*, \sigma(x_{-n}^*), y^*) \le J_n^F(x_n, \sigma(x_{-n}^*), y^*) \\ &\forall y \in \mathcal{Y} : J^L(y^*, \sigma_0(x_{\mathcal{N}}^*)) \le J^L(y, \sigma_0(x_{\mathcal{N}}^*)) \end{aligned} \tag{6}$$

$\forall n \in \mathcal{N}$ where $x_{-n}^* = \mathbf{col}(x_1^*, \dots, x_{n-1}^*, x_{n+1}^*, \dots, x_N^*)$ and $x_{\mathcal{N}}^* = \mathbf{col}(x_1^*, \dots, x_N^*)$. □

Equation (6) means that neither leader nor followers can improve their cost by unilateral deviation from SE point.

III. Leader–Follower Network Game

A. Communication and Information Structure

1) Followers' Communication and Activeness. We consider that follower n receives information of its neighbor $m \in \mathcal{N}_n$ at iteration k with probability p^k_{mn}. Furthermore, the follower n is active at iteration k to update its decision with probability q^k_n. Let random binary variables $l^k_{n,m}$ and e^k_n denote the establishment of communication from follower n to m, and activeness of follower n at iteration k, respectively. Clearly, $\forall k \geq 0 : l^k_{n,m} = 0$ for nonneighbor followers ($a_{nm} = 0$) and e^k_n is equal to 0 or 1 for inactive or active follower n at iteration k, respectively. Therefore, the last information of follower n from follower m denoted by $\tilde{x}^k_{n,m}$ is updated as follows:

$$\tilde{x}^{k+1}_{n,m} = (1 - l^k_{n,m})\tilde{x}^k_{n,m} + l^k_{n,m} x^k_m. \tag{7}$$

Based on (7), the aggregated strategy of neighborhoods of follower n at iteration k is calculated as $\tilde{\sigma}^k_n = \sigma_n(\tilde{x}^k_{n,-n})$, where $\tilde{x}^k_{n,-n} = \mathbf{col}(\tilde{x}^k_{n,1}, \ldots, \tilde{x}^k_{n,n-1}, \tilde{x}^k_{n,n+1}, \ldots, \tilde{x}^k_{n,N})$.

Let's consider $\mathcal{L}^k = [l_{nm}]_{n,m\in\mathcal{N}}$ and $\mathcal{E}^k = \mathbf{col}(e^k_1, \ldots, e^k_N)$ as the connectivity matrix and the activity vector of the followers at iteration k, respectively. The constraint set $\mathcal{P}$ represents the set from which $\mathcal{L}^k$ and $\mathcal{E}^k$ are selected for all iterations $\forall k \geq 0$. Further, $\mathcal{H}^k$ denotes the history set of stochastic variables $\mathcal{L}^k$ and $\mathcal{E}^k$ up to iteration k which is defined by $\mathcal{H}^{k+1} = \mathcal{H}^k \cup \{(\mathcal{L}^k, \mathcal{E}^k)\}$ and $\mathcal{H}^1 = \{(\mathcal{L}^0, \mathcal{E}^0)\}$. In this article, the probabilities of $\mathcal{L}^k$ and $\mathcal{E}^k$ are considered to be possibly dependent to $\mathcal{H}^k$ and $\mathcal{P}$ as follows:

$$\mathbf{P}\{l^k_{nm} = 1|\mathcal{F}^k\} = p^k_{nm}, \mathbf{P}\{e^k_n = 1|\mathcal{F}^k\} = q^k_n \tag{8}$$

where $\mathcal{F}^k = \mathcal{H}^k \cap \mathcal{P}$. Clearly, $\mathcal{F}_k \subset \mathcal{F}_{k+1}$ holds for $\forall k \geq 0$. To illustrate the dependency among stochastic variables, in what follows, the well-known Gossip-based communication protocol [21] has been studied as an example of the proposed communication framework.

Example 1 (Gossip-Based Communication): Suppose that the followers communicate to each other through an undirected graph $G(\mathcal{N}, \mathcal{A})$. In gossip-based communication, at most one pair of agents wake up at each time slot. For instance, at kth time slot, the agent n wakes up, and contacts with only one neighbor (say agent m). They communicate with each other, update their strategies and then both go to sleep. In this case, our communication framework imposes the constraints $\mathcal{P} = \{(\mathcal{L}^k, \mathcal{E}^k) \big| \sum_{n\in\mathcal{N}} e^k_n = 2, l^k_{nm} = e^k_n e^k_m, e^k_n e^k_m \leq a_{nm}, \forall n, m \in \mathcal{N}\}$. The first constraint indicates that only two player could be active in iteration k. $l^k_{nm} = e^k_n e^k_m$ implies that a link is established when both of its sender and receiver are active. Furthermore, inequality constraint $e^k_n e^k_m \leq a_{nm}$ prevents two nonneighbors to become active. The probability of each link and each node are calculated as $p^k_{nm} = \frac{1}{N}(\frac{1}{|\mathcal{N}_n|} + \frac{1}{|\mathcal{N}_m|})$ and $q^k_n = \frac{1}{N}(1 + \sum_{m\in\mathcal{N}_n} \frac{1}{|\mathcal{N}_m|})$ where $\mathcal{N}_n = \{m|a_{mn} = 1\}$. □

Assumption 2: There exist $\gamma > 0$ and $\delta > 0$ such that $p_{nm} \geq \gamma$ and $q_n \geq \delta$ for $\forall n \in \mathcal{N}, \forall m \in \mathcal{N}_m$. □

Note that $p_{nm} = 0$ for $\forall n \in \mathcal{N}, \forall m \notin \mathcal{N}_m$.

2) Communication between leader and followers: It is assumed that the leader does not exchange information with the followers at every iterations. Instead, it is considered that the leader infinitely often wakes up and receives aggregated strategy of the followers in an arbitrary desired iteration set $\mathcal{K}^L = \{k^L_i\}^\infty_{i=0}$. The leader also updates its decision variable and broadcasts it to all the followers at the same iteration. It is supposed that $k^L_0 = 0$.

Assumption 3: There is $\bar{K} < \infty$ such that $k^L_{i+1} - k^L_i \leq \bar{K}$ for $\forall i \in \mathbb{N}$. □

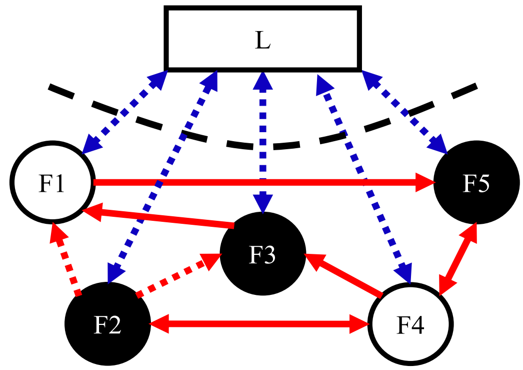

Fig. 1. Information scheme of leader–follower NAG between followers (F) and the leader (L). The filled and unfilled objects indicate the active and inactive agents, respectively. The dashed lines indicate the unestablished communication lines at an iteration.

The schematic of information flows among followers and between followers and leader is shown in Fig. 1. Each information exchange among followers could be established stochastically $\forall n, m \in \mathcal{N}$.

B. Decision Making

In this article, the projected subgradient method is utilized for decision making of each follower as follows:

$$x^{k+1}_n = \Pi_{\mathcal{X}_n}(x^k_n - e^k_n \alpha^k_n g^k_n) \tag{9}$$

where $g^k_n = d_n(x^k_n, \tilde{\sigma}^k_n, y^k)$ and α^k_n is the step size of follower n at iteration k. Clearly, x^k_n is updated only if the follower n is active at iteration k, i.e., $e^k_n = 1$. Also, the leader updates its strategy at $k \in \mathcal{K}^L$ as follows:

$$y^{k+1} = \Pi_{\mathcal{Y}}(y^k - \alpha^k_0 g^k_0);\ \forall k \in \mathcal{K}^L$$
$$y^{k+1} = y^k;\ \forall k \notin \mathcal{K}^L \tag{10}$$

where $g^k_0 = d_0(y^k, \sigma^k_0)$ and α^k_0 is the step size of the leader at iteration k. Without loss of the generality, we set $\alpha^{k+1}_0 = \alpha^k_0, \forall k \notin \mathcal{K}^L$. In this article, the following assumptions are considered for the step sizes of the players.

Assumption 4: $\alpha^k_n, \forall n \in \mathcal{N} \cup \{0\}$ are nonincreasing, $\sum^\infty_{k=0} \alpha^k_n = \infty$, and $\sum^\infty_{k=0}(\alpha^k_n)^2 < \infty$. □

Assumption 5: There exists κ such that $\overline{\alpha}^k \leq \kappa \underline{\alpha}^k$, where $\overline{\alpha}^k = \max(\alpha^k_1, \ldots, \alpha^k_N, \alpha^k_0)$ and $\underline{\alpha}^k = \min(\alpha^k_1, \ldots, \alpha^k_N, \alpha^k_0)$. □

The optimization procedure for the leader–follower network game is presented in Algorithm 1. Based on Algorithm 1, the leader makes decision at iterations $k^L_i \in \mathcal{K}^L$ and waits in other iterations, while the followers are making decision. In other words, the followers continue their interactions and decision makings based on the last informed decision of the leader for some iterations, until the next decision of the leader is announced. The initial values of the parameters are chosen from their feasible region.

IV. Convergence Analysis

In this section, the convergence of Algorithm 1 to the unique SE point of $\mathcal{G}$ is studied. Under Assumption 1, as a result of strong convexity of the cost functions, there exists a SE point $z^* = (x^*_{\mathcal{N}}, y^*)$ for $\mathcal{G}$ [22]. The convergence of Algorithm 1 is proven in Theorem 1.

Algorithm 1 The Leader-Follower Network Game Algorithm.

Initialize x_n, y and $\tilde{x}_{nm}$ for $\forall n, m \in \mathcal{N}$ and $k \leftarrow 0$
Iteration
  Leader:
    $\sigma_0 \leftarrow \sigma_0(x_{\mathcal{N}})$
    $g_0 \leftarrow d_0(y, \sigma_0)$
    $y \leftarrow \Pi_{\mathcal{Y}}(y - \alpha_0^k g_0))$
  Repeat
    Follower $n \in \mathcal{N}$:
      If $e_n^k = 1$:
        $\tilde{\sigma}_n \leftarrow \sigma_n(\tilde{x}_{n,-n})$
        $g_n \leftarrow d_n(x_n, \tilde{\sigma}_n, y)$
        $x_n \leftarrow \Pi_{\mathcal{X}_n}(x_n - \alpha_n^k g_n)$
      update $\tilde{x}_{nm}$ via (7) based on l_{nm}^k
    $k \leftarrow k+1$
  Until $k \in \mathcal{K}^L$

Theorem 1: Consider Assumptions 1, 2, 3, 4, and 5. If the constants C_n and C_0 in (4) satisfy $C_n > \frac{\kappa}{\delta}\bar{L}$ and $C_0 > \kappa \bar{K} \bar{L}$ $\forall n \in \mathcal{N}$, Algorithm 1 almost surly converges to the SE point of the leader–follower game where $\bar{L} = \max(2\,L, L_0)$. □

Proof: Consider the notation $\nabla x_n^k = x_n^k - x_n^*$. Based on [23, Proposition 1.5.8], $x_n^* = \Pi_{\mathcal{X}_n}(x_n^* - e_n^k \alpha_n^k d_n(x_n^*, \sigma_n^*, y^*))$ where $\sigma_n^* = \sigma_n(x_{-n}^*)$. Since the projection operator $\Pi_{\mathcal{X}_n}(.)$ is nonexpansive and $e_n^k \leq 1$, we have

$$\begin{aligned}||\nabla x_n^{k+1}||^2 &\leq ||\nabla x_n^k - e_n^k \alpha_n^k \left(d_n(x_n^k, \tilde{\sigma}_n^k, y^k)\right.\\ &\quad \left.- d_n(x_n^*, \sigma_n^*, y^*)\right)||^2 = ||\nabla x_n^k||^2\\ &\quad + (e_n^k \alpha_n^k)^2 ||d_n(x_n^k, \tilde{\sigma}_n^k, y^k) - d_n(x_n^*, \sigma_n^*, y^*)||^2\\ &\quad - 2 e_n^k \alpha_n^k \left(d_n(x_n^k, \tilde{\sigma}_n^k, y^k) - d_n(x_n^*, \sigma_n^*, y^*)\right)^\top \nabla x_n^k\\ &\leq ||\nabla x_n^k||^2 + 4A_n^2(\alpha_n^k)^2 - 2e_n^k\alpha_n^k\Psi_n^k - 2e_n^k\alpha_n^k\Omega_n^k \end{aligned} \tag{11}$$

where $\Psi_n^k = (d_n(x_n^k, \sigma_n^k, y^k) - d_n(x_n^*, \sigma_n^*, y^*))^\top \nabla x_n^k$, $\Omega_n^k = (d_n(x_n^k, \tilde{\sigma}_n^k, y^k) - d_n(x_n^k, \sigma_n^k, y^k))^\top \nabla x_n^k$, and $\sigma_n^k = \sigma_n(x_{-n}^k)$. Let consider that $||x_1 - x_2|| \leq B_n$ for $\forall x_1, x_2 \in \mathcal{X}_n$. According to Assumption 1, we have

$$\begin{aligned}- e_n^k \alpha_n^k \Omega_n^k &\leq \alpha_n^k L ||\tilde{\sigma}_n^k - \sigma_n^k|| ||\nabla x_n^k||\\ &\leq L B_n \alpha_n^k ||\tilde{\sigma}_n^k - \sigma_n^k|| \leq L B_n \alpha_n^k \sum_{m \in \mathcal{N}_n} w_{nm} ||\Delta \tilde{x}_{nm}^k||.\end{aligned} \tag{12}$$

Therefore, by putting (12) into (11), we have

$$\begin{aligned}||\nabla x_n^{k+1}||^2 &\leq ||\nabla x_n^k||^2 + 4A_n^2(\alpha_n^k)^2\\ &\quad - 2e_n^k\alpha_n^k\Psi_n^k + 2\,L B_n \alpha_n^k \sum_{m \in \mathcal{N}_n} w_{nm}||\Delta\tilde{x}_{nm}^k||.\end{aligned} \tag{13}$$

Suppose the notation $\nabla y^k = y^k - y^*$. Considering the leader's decision at leader's iteration k_j^L, and following the same operation from (11) to (13), we have

$$||\nabla y^{k_j^L+1}||^2 \leq ||\nabla y^{k_j^L}||^2 + 4(\alpha_0^{k_j^L})^2 A_0^2 - 2\alpha_0^{k_j^L}\Psi_0^{k_j^L}. \tag{14}$$

where $\Psi_0^k = (d_0(y^k, \sigma_0^k) - d_0(y^*, \sigma_0^*))^\top \nabla y^k$, $\sigma_0^k = \sigma_0(x_{\mathcal{N}}^k)$, $\sigma_0^* = \sigma_0(x_{\mathcal{N}}^*)$ and $||d_0(y, \sigma_0)|| \leq A_0$. Now, let define $\Phi^j = ||\nabla y^{k_{j-1}^L+1}||^2 + \sum_{n \in \mathcal{N}} ||\nabla x_n^{k_{j-1}^L+1}||^2$. Therefore, using inequalities (13) and (14) for $k = k_{j-1}^L, \ldots, k_j^L$, we have:

$$\begin{aligned}\mathbf{E}\{\Phi^{j+1} \big| \mathcal{F}^{k_j^L}\} &\leq \Phi^j + 4(\alpha_0^{k_j^L})^2 A_0^2 + 4\sum_{n\in\mathcal{N}} A_n^2 \sum_{k' \in \mathcal{K}_j'} (\alpha_n^{k'})^2\\ &\quad + 2\,L \sum_{n \in \mathcal{N}} \sum_{m \in \mathcal{N}_n} B_n w_{nm} \sum_{k' \in \mathcal{K}_j'} \alpha_n^{k'} ||\Delta \tilde{x}_{nm}^{k'}||\\ &\quad - 2\alpha_0^{k_j^L}\Psi_0^{k_j^L} - 2\sum_{n\in\mathcal{N}}\sum_{k'\in\mathcal{K}_j'} \alpha_n^{k'} \mathbf{E}\{e_n^{k'} \big| \mathcal{F}^{k'}\}\Psi_n^{k'}.\end{aligned} \tag{15}$$

where $\mathcal{K}_j' = \{k_{j-1}^L + 1, \ldots, k_j^L\}$. Based on definition of $\Psi_n^{k'}$, and adding and subtracting the term $d_n(x_n^*, \sigma_n^{k'}, y^{k'})$, we have

$$\begin{aligned}\Psi_n^{k'} &= \left(d_n(x_n^{k'}, \sigma_n^{k'}, y^{k'}) - d_n(x_n^*, \sigma_n^{k'}, y^{k'})\right)^\top \nabla x_n^{k'}\\ &\quad + \left(d_n(x_n^*, \sigma_n^{k'}, y^{k'}) - d_n(x_n^*, \sigma_n^*, y^*)\right)^\top \nabla x_n^{k'}.\end{aligned} \tag{16}$$

Based on strongly convexity and Lipschitz property in Assumption 4, and using Assumptions 5, we have

$$\begin{aligned}-\alpha_n^{k'} e_n^{k'} \Psi_n^{k'} &\leq -\alpha_n^{k'} e_n^{k'} C_n ||\nabla x_n^{k'}||^2 + \alpha_n^{k'} L \Bigg(||\nabla y^{k_i^L}||\\ &\quad + ||\sigma_0^{k_i^L} - \sigma_0^*||\Bigg) ||\nabla x_n^{k'}|| \leq -\underline{\alpha}^{k'} C_n e_n^{k'} ||\nabla x_n^{k'}||^2\\ &\quad + \overline{\alpha}^{k'} \kappa L ||\nabla x_n^{k'}|| \Bigg(||\nabla y^{k'}|| + \sum_{m \in \mathcal{N}_n} w_{nm} ||\nabla x_m^{k'}||\Bigg).\end{aligned} \tag{17}$$

By following the same procedure for the leader, we have

$$\begin{aligned}-\alpha_0^{k_j^L}\Psi_0^{k_j^L} &\leq -\underline{\alpha}^{k_j^L} C_0 ||\nabla y^{k_j^L}||^2\\ &\quad + \overline{\alpha}^{k_j^L} L_0 \sum_{n \in \mathcal{N}} w_{0n} ||\nabla x_n^{k_j^L}|| ||\nabla y^{k_j^L}||.\end{aligned} \tag{18}$$

Considering Assumption 2, it is clear that $\mathbf{E}\{e_n^{k'} \big| \mathcal{F}^{k'}\} = q_n^k \geq \delta$. Therefore, by applying inequalities (17) and (18) into the last two terms of (15), it can be concluded that

$$\begin{aligned}&-\underline{\alpha}^{k_j^L}\Psi_0^{k_j^L} - \sum_{n\in\mathcal{N}}\sum_{k'\in\mathcal{K}_j'} \alpha_n^{k'} \mathbf{E}\{e_n^{k'} \big| \mathcal{F}^{k'}\}\Psi_n^{k'}\\ &\leq -\underline{\alpha}^{k_j^L} C_0 ||\nabla y^{k_j^L}||^2 - \sum_{n\in\mathcal{N}}\sum_{k'\in\mathcal{K}_j'} \delta \underline{\alpha}^{k'} C_n ||\nabla x_n^{k'}||^2\\ &\quad + \sum_{k'\in\mathcal{K}_j'} \overline{\alpha}^{k'} v^{k'\top} \mathcal{V}^{k'} v^{k'}.\end{aligned} \tag{19}$$

where $v^{k'} = \mathbf{col}(||\nabla x_1^{k'}||, \ldots, ||\nabla x_N^{k'}||, ||\nabla y^{k'}||)$, $\mathcal{V}^{k'} = \left[\begin{smallmatrix} L\mathcal{W} & L\vec{1}_N \\ L_0 \vec{w}_0^\top & 0 \end{smallmatrix}\right]$ for $k' \in \mathcal{K}^L$, and $\mathcal{V}^{k'} = \left[\begin{smallmatrix} L\mathcal{W} & L\vec{1}_N \\ \vec{0}_N^\top & 0 \end{smallmatrix}\right]$ for $k' \notin \mathcal{K}^L$. It is straightforward to see that the summation of each row of the matrices $[\,L\mathcal{W}\ \ L\vec{1}_N\,]$ and $[\,L_0\vec{w}_0^\top\ \ 0\,]$ are equal to $2\,L$ and L_0, respectively. Therefore, based on Perron–Frobenius Theorem [24], $||\mathcal{V}^{k'}|| \leq \max(2\,L, L_0)$ and $||\mathcal{V}^{k'}|| \leq 2\,L$ for $k' \in \mathcal{K}^L$ and $k' \notin \mathcal{K}^L$, respectively. Hence, $v^{k'\top} V^{k'} v^{k'} \leq \bar{L}||v^{k'}||^2$ for $\forall k' \geq 0$. Therefore,

$$\begin{aligned}\sum_{k'\in\mathcal{K}_j'} \underline{\alpha}^{k'} v^{k'\top}\mathcal{V}^{k'} v^{k'} &\leq \sum_{k'\in\mathcal{K}_j'} \underline{\alpha}^{k'} \bar{L} ||\nabla x_n^{k'}||^2\\ &\quad + \underline{\alpha}^{k_j^L}(k_j^L - k_{j-1}^L)\bar{L}||\nabla y^{k_j^L}||^2.\end{aligned} \tag{20}$$

where the last term of (20) is rearranged, since the leader only makes decision at $k_j^L \in \mathcal{K}^L$ and therefore, the term $||\nabla y^{k_j^L}||^2$ is the same from iteration k_{j-1}^L to k_j^L. By putting (19) and (20) into (15), it can be concluded that

$$\begin{aligned}\mathbf{E}\{\Phi^{j+1}|\mathcal{F}^{k_j^L}\} \leq \Phi^j &+ \underbrace{4(\alpha_0^{k_j^L})^2 A_0^2 + 4\sum_{n\in\mathcal{N}} A_n^2 \sum_{k'\in\mathcal{K}'_j} (\alpha_n^{k'})^2}_{T_1^j} \\ &+ \underbrace{2L\sum_{n\in\mathcal{N}}\sum_{m\in\mathcal{N}_n} B_n w_{nm} \sum_{k'\in\mathcal{K}'_j} \alpha_n^{k'} ||\Delta\tilde{x}_{nm}^{k'}||}_{T_2^j} \\ &- \underbrace{2\underline{\alpha}^{k_j^L}(C_0 - \kappa(k_j^L - k_{j-1}^L)\bar{L})||\nabla y^{k_j^L}||^2}_{T_3^j} \\ &- \underbrace{2\sum_{n\in\mathcal{N}}\sum_{k'\in\mathcal{K}'_j} \underline{\alpha}^{k'}(\delta C_n - \kappa\bar{L})||\nabla x_n^{k'}||^2}_{T_4^j}. \qquad (21)\end{aligned}$$

Based on Assumption 4, $\sum_{j=0}^{\infty} T_1^j$ is bounded. Also, based on Lemma 3 (see Appendix), we have $\sum_{j=0}^{\infty} T_1^j < \infty$. According to Assumption 3, $C_0 - (k_j^L - k_{j-1}^L)\kappa\bar{L} \geq C_0 - \bar{K}\kappa\bar{L} > 0$ and $\delta C_n - \kappa\bar{L} > 0$. Therefore, T_3^j and T_4^j are positive. Now, the assumptions of Lemma 1 (see Appendix) are satisfied and as a result, and consequently, $\sum_{j=0}^{\infty} T_3^j + T_4^j < \infty$ converges almost surely. Because of positiveness of T_3^j and T_4^j, $\sum_{j=0}^{\infty} T_3^j < \infty$ and $\sum_{j=0}^{\infty} T_4^j < \infty$ are concluded. Therefore, based on $\sum_{k=0}^{\infty} \underline{\alpha}^k \geq \frac{1}{\kappa}\sum_{k=0}^{\infty} \overline{\alpha}^k = \infty$, both $||\nabla x_n^{k'}||^2$ and $||\nabla y^{k_i^L}||^2$ almost surely converge to 0. Thus, x_n^k and y^k converge almost surely to x_n^* and y^*. ■

Theorem 1 results almost sure convergence of Algorithm 1. Nevertheless, almost sure convergence does not generally lead to mean square convergence. Corollary 1 proves the convergence of Algorithm 1 in mean square sense.

Corollary 1: Under Assumptions of Theorem 1, Algorithm 1 converges in mean square sense. □

Proof: Since almost sure convergence results the convergence in distribution, the expectation of $\sum_{j=0}^{\infty} T_3^j < \infty$ and $\sum_{j=0}^{\infty} T_4^j < \infty$ converges. Therefore, $\sum_{j=0}^{\infty} \underline{\alpha}^{k_j^L}\mathbf{E}\{||\nabla y^{k_j^L}||^2\} < \infty$ and $\sum_{k=0}^{\infty} \underline{\alpha}^k \mathbf{E}\{||\nabla x_n^k||^2\} < \infty$. Hence, because of $\sum_{k=0}^{\infty} \underline{\alpha}^k > \infty$, $\mathbf{E}\{||\nabla y^{k_j^L}||^2\}$ and $\mathbf{E}\{||\nabla x_n^k||^2\}$ converges to 0 that means that x_n^k and y^k converge in mean square to x_n^* and y^*. ■

In Theorem 1, the convergence of the algorithm to a SE point is studied. In the following proposition, the uniqueness of the SE point is proven.

Proposition 1: Under the assumptions of Theorem 1, the game (3) has a unique SE. □

Proof: Let define $z = \mathbf{col}(x_1, \ldots, x_N, y)$ and $z' = \mathbf{col}(x'_1, \ldots, x'_N, y')$. Also, consider the function $g(z) = \mathbf{col}(d_1(x_1, \sigma_1(x_{\mathcal{N}}), y), \ldots, d_N(x_N, \sigma_N(x_{\mathcal{N}}), y), d_0(y, \sigma_0(x_{\mathcal{N}})))$. Therefore, by following the procedure of (17) and (18), we have

$$\begin{aligned}\Psi &= (z - z')^\top (g(z) - g(z')) \\ &= (y - y')^\top (d_0(y, \sigma_0(x_{\mathcal{N}})) - d_0(y', \sigma_0(x'_{\mathcal{N}})) \\ &\quad + \sum_{n\in\mathcal{N}} (x_n - x'_n)^\top (d_n(x_n, \sigma_1(x_{\mathcal{N}}), y) - d_n(x'_n, \sigma_1(x'_{\mathcal{N}}), y')) \\ &\geq C_0||\nabla y||^2 - L_0 \sum_{n\in\mathcal{N}} w_{0n}||\nabla x_n|| ||\nabla y|| \\ &\quad + \sum_{n\in\mathcal{N}} C_n||\nabla x_n||^2 - L\left(||\nabla y|| + \sum_{m\in\mathcal{N}_n} w_{nm}||\nabla x_m||\right)||\nabla x_n|| \end{aligned} \qquad (22)$$

where $\nabla x_n = x_n - x'_n$ and $\nabla y = y - y'$. Considering $v = \mathbf{col}(||\nabla x_1||, \ldots, ||\nabla x_N||, ||\nabla y||)$, it can be concluded that $\Psi \geq C_0||\nabla y||^2 + \sum_{n\in\mathcal{N}} C_n||\nabla x_n||^2 - v^\top \mathcal{R} v$ where $\mathcal{R} = [\begin{smallmatrix} L\mathcal{W} & L\bar{1}_N \\ L_0\bar{w}_0^\top & 0 \end{smallmatrix}]$. Based on Perron–Frobenius Theorem, $v^\top \mathcal{R} v \leq \bar{L}||v||^2$. Also, it is clear that $||v||^2 = ||\nabla y||^2 + \sum_{n\in\mathcal{N}} ||\nabla x_n||^2$. Therefore, it can be deduced that $\Psi \geq (C_0 - \bar{L})||\nabla y||^2 + \sum_{n\in\mathcal{N}} (C_n - \bar{L})||\nabla x_n||^2$. Based on the assumptions of Theorem 1, $C_n \geq \frac{\kappa}{\delta}\bar{L} > \bar{L}$ and $C_0 > \kappa\bar{K}\bar{L} \geq \bar{L}$ because $\kappa, \bar{K} \geq 1$ and $\delta \leq 1$. Hence, $g(z)$ is strictly monotone since $\Psi > 0$. Consequently, according to [22, Th. 2], the SE of the game (3) is unique. ■

V. Simulation Results

As an application of leader–follower NAG, we study the power allocation of small cell networks proposed in [6]. Consider a network consisting of N small cells, all of which underlay a macrocell with a macrocell base station (MBS). Small cells and macrocell provide radio coverage for cellular networks. However, small cells are low-power and have limited coverage range in comparison with the macrocell. Each small cell is considered to have a small cell base station (SBS), which can cover many users. Deployment of multiple SBS in a region may cause some overlapping coverage region among SBSs. In such region, transmission powers of SBSs cause the signal interference in cellular networks. As a result, based on Shanon formula utilized in (23), the data rate will be decreased. In this case, the interference appears as an aggregative term in the cost function of neighboring SBSs and therefore, the problem can be modeled as a NAG. In this NAG, each SBS aims to adjust its transmission power to minimize its cost. Moreover, MBS, as a leader of small cells' network, determines the price of transmission power for SBSs as its decision variable. Consequently, there is a leader–follower NAG among the SBSs and MBS. Let $\mathcal{N}$ denotes the set of small cells. x_n denotes the power of SBS $n \in \mathcal{N}$, which satisfies $0 \leq x_n \leq \bar{P}_n$. The objective function of SBS $n \in \mathcal{N}$ is as follows:

$$\begin{aligned} &J_n(x_n, x_{-n}, \lambda) = R_n(S_n) - \lambda v_n x_n \\ &R_n(S_n) = ALn(1 + S_n) \\ &S_n = \frac{r_n^{-\beta} x_n}{N_0 + \sum_{m\in\mathcal{N}_n} r_{nm}^{-\beta} x_m}, v_n = \sum_{n\in\mathcal{N}} r_{nm}^{-\beta} \end{aligned} \qquad (23)$$

where S_n and $R_n(S_n)$ indicate signal to interference and noise ratio and the data rate corresponding to the SBS n, respectively. A is the channel bandwidth. SBS n has the strategy x_n. r_n and r_{nm} are the average distance of SBS n to its users and the distance of SBS n to SBS m, respectively. Based on the coverage range, SBS n could be interfered from a set of other SBSs indicated by $\mathcal{N}_n$. β is the path-loss exponent and N_0 is the white noise spectral density. The penalty term $\lambda v_n x_n$ specifies the cost for making interference to other SBSs in the network where, λ is the penalty price. The objective function of the MBS is as follows:

$$J_0(\lambda, x_{\mathcal{N}}) = \lambda \sum_{n\in\mathcal{N}} v_n x_n - B_0\lambda^2 \qquad (24)$$

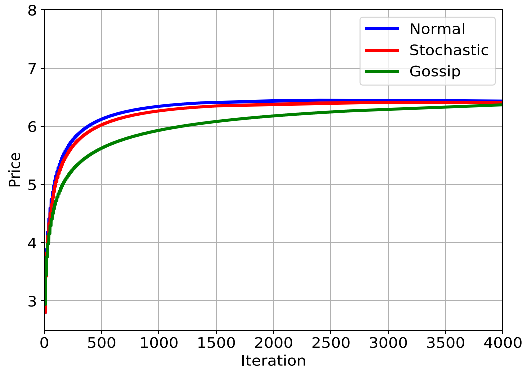

Fig. 2. Price of leader along the algorithm iterations.

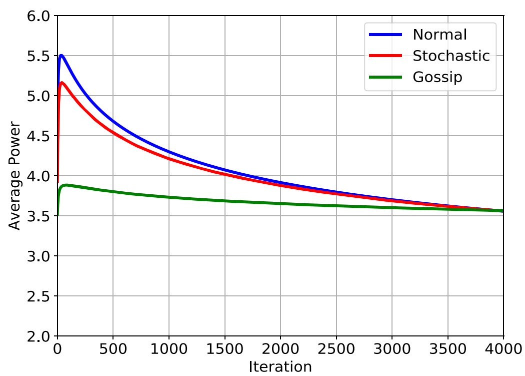

Fig. 3. Average power of followers along the algorithm iterations.

λ satisfies $0 \leq \lambda \leq \bar{\lambda}$. The term $B_0\lambda^2$ prevents MBS to increase λ too much. Based on the proposed framework of the game, the MBS and SBSs can be considered as a leader and followers, respectively.

For simulation, ten SBSs are considered to be stochastically located in a circular region with radius 4 km. It is assumed that each of two SBSs with less than 1 km distances are neighbors and have an interference effect on each other. Also, $A = 2048$ bps, $\bar{P}_n = 6$w, $\beta = 1$, for $\forall n \in \mathcal{N}$. Also, $\bar{\lambda} = 7$ and $B_0 = 100$. We assume that MBS (the leader) makes decisions periodically once at every ten iterations. The simulation is done for three communication protocols; 1) Normal ($p^k_{nm} = q^k_n = 1$ for $\forall n \in \mathcal{N}, m \in \mathcal{N}_n, \forall k \geq 0$) 2) Stochastic ($p^k_{nm} = q^k_n = 0.7$ for $\forall n \in \mathcal{N}, m \in \mathcal{N}_n, \forall k \geq 0$) 3) Gossip. The results for the leader's price and average power of the followers are shown in Figs. 2 and 3, respectively. As it can be seen from Fig. 2, the price diagram is a piecewise-constant signal based on periodic iterations of the leader. Also, as shown in Fig. 3, the price converges slowly in gossip-based protocol compared to two other protocols, since just two of followers communicate and update at each iteration. Clearly, the lesser the number of updates, the slower the progress of optimization toward the equilibrium point for the followers. The stochastic scenario has an acceptable performance in comparison with normal scenario, but with lesser active agents. Therefore, the agents can economically communicate with each other and update their decisions.

VI. Conclusion

This article proposed a leader–follower scheme for NAGs, considering stochastic activeness of the followers and also stochastic communication among them. In particular, the aim was to find the noncooperative equilibrium point of the game. A distributed algorithm was proposed, and its convergence to the unique SE point of the game was proven in both mean square and almost sure senses. To prove the convergence of algorithm, we imposed the assumption of strong convexity to the cost functions. There are some methods in the literature such as Tikhonov regularization and proximal point [25], [26], which can handle the optimization problem with lower level of convexity. As a future work, one can explore such methods to find a more relaxed condition on the cost function of the leader and followers.

Appendix

Lemma 1 (Theorem 1 of [27]): Let z_k, β_k, η_k, and ζ_k be nonnegative $\mathcal{F}_k$-measurable random variables. Also, assume that $\mathcal{F}_k$ is σ-algebra and $\mathcal{F}_k \subset \mathcal{F}_{k+1}$ holds for $\forall k \geq 0$. If $\sum_{k=0}^{\infty} \beta_k$ and $\sum_{k=0}^{\infty} \eta_k$ almost surely converge, and the following equation:

$$\mathbf{E}\{z_{k+1}|\mathcal{F}_k\} \leq (1+\beta_k)z_k + \eta_k - \zeta_k \tag{25}$$

holds, then z_k and $\sum_{k=0}^{\infty} \zeta_k < \infty$ almost surely converge. □

Lemma 2: $||x^{k+1}_n - x^k_n|| \leq A_n \alpha^k_n$ for $\forall n, m \in \mathcal{N}$ where $||d_n(x_n, \sigma, y)|| \leq A_n, \forall x_n \in \mathcal{X}_n$. □

Proof: Based on (9) and $e^k_n \leq 1$, we have

$$\begin{aligned}||x^{k+1}_n - x^k_n|| &= ||\Pi_{\mathcal{X}_n}(x^k_n - e^k_n\alpha^k_n d_n(x^k_n, \tilde{\sigma}^k_n, y^k)) - x^k_n|| \\ &\leq e^k_n\alpha^k_n||d_n(x^k_n, \tilde{\sigma}^k_n, y^k)|| \leq \alpha^k_n A_n.\end{aligned}$$

■

Lemma 3: $\sum_{k=0}^{\infty} \alpha^k_n||\Delta\tilde{x}^k_{nm}|| < \infty$ for $\forall n, m \in \mathcal{N}$ in almost sure sense where $\Delta\tilde{x}^k_{nm} = \tilde{x}^k_{nm} - x^k_m$. □

Proof: From (7) and Lemma 2, it can be deduced that

$$\begin{aligned}\mathbf{E}\{||\Delta\tilde{x}^{k+1}_{nm}|||\mathcal{F}^k\} &= (1 - \mathbf{E}\{l^k_{nm}|\mathcal{F}^k\})||\tilde{x}^k_{nm} - x^{k+1}_m|| \\ &= (1 - p^k_{nm})||(\tilde{x}^k_{nm} - x^k_m) - (x^{k+1}_m - x^k_m)|| \\ &\leq (1-\gamma)(||\Delta\tilde{x}^k_{nm}|| + \alpha^k_m A_m).\end{aligned} \tag{26}$$

From Assumption 4, α^k_n is nonincreasing, i.e., $\alpha^{k+1}_n \leq \alpha^k_n$. Therefore, by multiplying $\alpha^{k+1}_n \leq \alpha^k_n$ and (26), we have

$$\begin{aligned}&\mathbf{E}\{\alpha^{k+1}_n||\Delta\tilde{x}^{k+1}_{nm}|||\mathcal{F}^k\} \\ &\leq (1-\gamma)(\alpha^k_n||\Delta\tilde{x}^k_{nm}|| + \alpha^k_n\alpha^k_m A_m) \\ &= \alpha^k_n||\Delta\tilde{x}^k_{nm}|| - \gamma\alpha^k_n||\Delta\tilde{x}^k_{nm}|| + (1-\gamma)\alpha^k_n\alpha^k_m A_m.\end{aligned} \tag{27}$$

Based on Assumption 4, it is straightforward that $\sum_{k=1}^{\infty} \alpha^k_p\alpha^k_m < \infty$. Consequently, the assumptions of Lemma 1 are satisfied in (27). As a result, $\sum_{k=1}^{\infty} \alpha^k_n||\Delta\tilde{x}^k_{nm}|| < \infty$. ■

References

[1] V. S. Mai and E. H. Abed, "Distributed optimization over directed graphs with row stochasticity and constraint regularity," *Automatica*, vol. 102, pp. 94–104, 2019.
[2] F. Salehisadaghiani and L. Pavel, "Distributed Nash equilibrium seeking in networked graphical games," *Automatica*, vol. 87, pp. 17–24, 2018.
[3] A. Galeotti, S. Goyal, M. O. Jackson, F. Vega-Redondo, and L. Yariv, "Network Games," *Rev. Econ. Stud.*, vol. 77, no. 1, pp. 218–244, 2010.
[4] C. Zhao, U. Topcu, N. Li, and S. Low, "Design and stability of load-side primary frequency control in power systems," *IEEE Trans. Autom. Control*, vol. 59, no. 5, pp. 1177–1189, May 2014.
[5] S. Grammatico, "Opinion dynamics are proximal dynamics in multi-agent network games," in *Proc. IEEE 56th Annu. Conf. Decis. Control*, Dec. 2017, pp. 3835–3840.
[6] P. Semasinghe, E. Hossain, and S. Maghsudi, "Cheat-proof distributed power control in full-duplex small cell networks: A repeated game with imperfect public monitoring," *IEEE Trans. Commun.*, vol. 66, no. 4, pp. 1787–1802, Apr. 2018.

[7] N. Allouch, "On the private provision of public goods on networks," *J. Econ. Theory*, vol. 157, pp. 527–552, 2015.

[8] Y.-J. Chen, Y. Zenou, and J. Zhou, "Multiple activities in networks," *Amer. Econ. J.: Microecon.*, vol. 10, no. 3, pp. 34–85, Aug. 2018.

[9] S. Liang, P. Yi, and Y. Hong, "Distributed Nash equilibrium seeking for aggregative games with coupled constraints," *Automatica*, vol. 85, pp. 179–185, 2017. [Online]. Available: http://www.sciencedirect.com/science/article/pii/S0005109817304132

[10] J. Koshal, A. Nedić, and U. V. Shanbhag, "Distributed algorithms for aggregative games on graphs," *Oper. Res.*, vol. 64, no. 3, pp. 680–704, 2016.

[11] D. Paccagnan, B. Gentile, F. Parise, M. Kamgarpour, and J. Lygeros, "Nash and Wardrop equilibria in aggregative games with coupling constraints," *IEEE Trans. Autom. Control*, vol. 64, no. 4, pp. 1373–1388, Apr. 2019.

[12] F. Parise, B. Gentile, S. Grammatico, and J. Lygeros, "Network aggregative games: Distributed convergence to Nash equilibria," in *Proc. 54th IEEE Conf. Decis. Control*, Dec. 2015, pp. 2295–2300.

[13] S. Grammatico, "Proximal dynamics in multiagent network games," *IEEE Trans. Control Netw. Syst.*, vol. 5, no. 4, pp. 1707–1716, Dec. 2018.

[14] F. Parise and A. Ozdaglar, "A variational inequality framework for network games: Existence, uniqueness, convergence and sensitivity analysis," *Games Econ. Behav.*, vol. 114, pp. 47–82, 2019.

[15] M. Nourian, P. E. Caines, R. P. Malhame, and M. Huang, "Mean field LQG control in leader-follower stochastic multi-agent systems: Likelihood ratio based adaptation," *IEEE Trans. Autom. Control*, vol. 57, no. 11, pp. 2801–2816, Nov. 2012.

[16] H. Kebriaei and L. Iannelli, "Discrete-time robust hierarchical linear-quadratic dynamic games," *IEEE Trans. Autom. Control*, vol. 63, no. 3, pp. 902–909, Mar. 2018.

[17] B. Wang and J. Zhang, "Hierarchical mean field games for multiagent systems with tracking-type costs: Distributedε-Stackelberg equilibria," *IEEE Trans. Autom. Control*, vol. 59, no. 8, pp. 2241–2247, Aug. 2014.

[18] M. Saffar, H. Kebriaei, and D. Niyato, "Pricing and rate optimization of cloud radio access network using robust hierarchical dynamic game," *IEEE Trans. Wireless Commun.*, vol. 16, no. 11, pp. 7404–7418, Nov. 2017.

[19] Y. Yu, G. Q. Huang, and L. Liang, "Stackelberg game-theoretic model for optimizing advertising, pricing and inventory policies in vendor managed inventory (VMI) production supply chains," *Comput. Ind. Eng.*, vol. 57, no. 1, pp. 368–382, 2009.

[20] G. El Rahi, S. R. Etesami, W. Saad, N. B. Mandayam, and H. V. Poor, "Managing price uncertainty in prosumer-centric energy trading: A prospect-theoretic stackelberg game approach," *IEEE Trans. Smart Grid*, vol. 10, no. 1, pp. 702–713, Jan. 2019.

[21] S. Boyd, A. Ghosh, B. Prabhakar, and D. Shah, "Randomized gossip algorithms," *IEEE Trans. Inf. Theory*, vol. 52, no. 6, pp. 2508–2530, Jun. 2006.

[22] J. B. Rosen, "Existence and uniqueness of equilibrium points for concave n-person games," *Econometrica*, vol. 33, no. 3, pp. 520–534, 1965.

[23] J.-S. P. Francisco Facchinei, *Finite Dimensional Variational Inequalities and Complementarity Problems*, 1st ed. Berlin, Germany: Springer, 2003, vol. 1.

[24] T. C. Y. Cheng and M. Elgindi, "A note on the proof of the perron-Frobenius theorem," *Appl. Math.*, vol. 3, no. 11, pp. 1697–1701, 2012.

[25] P. Yi and L. Pavel, "Distributed generalized Nash equilibria computation of monotone games via double-layer preconditioned proximal-point algorithms," *IEEE Trans. Control Netw. Syst.*, vol. 6, no. 1, pp. 299–311, Mar. 2019.

[26] N. Buong, "Tikhonov regularization for a general nonlinear constrained optimization problem," *Comput. Math. Math. Phys.*, vol. 47, no. 10, pp. 1583–1588, Oct. 2007. [Online]. Available: https://doi.org/10.1134/S0965542507100016

[27] H. Robbins and D. Siegmund, "A convergence theorem for non negative almost supermartingales and some applications," in *Optimizing Methods in Statistics*, New York, NY, USA: Academic, 1971, pp. 233–257.