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# Stable multi-player evolutionary game for two-choice resource allocation problems

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**Abstract:** The authors consider the problem of multi-player evolutionary game with two strategies and an arbitrary number of players. A randomised probabilistic access model is considered where the users randomly select the strategies. The strategies are a selection between two radio access technologies to receive service from. For a fair resource allocation between a number of users, the payoff function of each player is considered to be inversely proportional to the number of players choosing a strategy. It is valid for the radio access technology selection problem where resources are divided fairly between the users. The necessary and sufficient conditions for the existence and uniqueness of a mixed Nash equilibrium point of the game is provided. In addition, it is proved that the mixed Nash equilibrium point of the game is globally asymptotically stable and also evolutionarily stable strategy.

## 1 Introduction

In recent years, by increasing the trend of research and developments on multi-agent systems, the application of game theory is demonstrated. Game theory has long been used to tackle modelling, analysis and decision making in resource allocation and other problems on a small scale [1–4]; however, for larger scale and more complex systems, the strategy dynamics are better modelled by evolutionary game theory [5, 6].

Evolutionary game theory was originally developed for biology applications [7, 8], but soon its applications were extended to other fields [9]. Most of the works around evolutionary game theory consider 2-player games [10–13] but some works consider situations where there is no pairwise interaction in competitions. Many applications of evolutionary games model inherent non-linearities. One way of introducing such non-linearities into the evolutionary game model is to include multiple players in evolutionary games[14]. Multi-player evolutionary games introduce models for situations where more than two individual models are competing simultaneously [15, 16].

Since the introduction of multi-player evolutionary games, there has been an ongoing research around it about its equilibrium points. In [17], a three players evolutionary game with two strategies for each player is considered the evolutionary and asymptotic stability is discussed. In [18], the authors choose aspiration method for deciding on the next action of players instead of mutation method. Although this way needs less information about the utility function, we choose the mutation method as we know and select our desirable utility function. The authors in [14, 19] study multi-player evolutionary game dynamics. The papers discuss the possible number and the probability of a certain number of internal equilibria points. Traulsen *et al.* [20] also address the probability of a certain number of equilibria points. The difference is in that it also addresses the effects of the number of players and the number of strategies and discuss the same problem when the payoff matrices are drawn randomly from an arbitrary distribution. Wu *et al.* [21] tries to solve the problem of equilibria points in a different manner. It considers the multi-player game as sets of two players games and tries to solve the problem by decomposing it into several smaller two players games. Riehl and Cao [22] discussed the equilibrium points in multi-player games in finite populations. Motro [23] considers the problem of equilibrium points in the case where all players gain a similar amount of resources regardless of the strategy they choose. Bukowski and Miekisz [24] investigate evolutionary and asymptotic stability for three players, two strategies game. Pena *et al.* [25] unify, simplify and extend the previous work on evolutionary multi-player game with two strategies and use observations to conclude equilibrium points and its properties. In the above-mentioned literature, the equilibrium points of the game have been considered in different condition and situation. Another problem can be studied by multi-player evolutionary game is resource allocation.

Allocating resources by optimising an objective function, in particular, a linear one is one of the classic problems in operations research which can be traced back to the foundation of linear programming [26, 27]. Studying this problem by classic approaches gives a static, fix results. In the other words, the results show which resource should be received by which user, specifically. Moreover, in these methods, the propose of optimisation focused on maximising the specific objective function. This point of view does not guarantee the effects of different users functioning. In the literature, there are different game theoretical resource allocation algorithms. One of the most common approaches is auction and bidding based methods [28–31], where the competition occurs between users and resource providers. In such games usually, users bid for the resources and the providers respond. Min-max and Stackelberg games are also utilised for this purpose [32, 33]. Moreover, in [34, 35] authors model the behaviour of providers as a cooperative games. These approaches focus on the resources to choose their way of sharing. In addition, [36–38] use a Monitory game to make users learn how to choose their resources in a way, not too overloaded the resources. Our model lets the users choose their intended resource independently and receives its proportion of resource according to the number of users choosing that resource. This model could be proper when the resources do not have any preference between the users to share their resource. Although all the models in literature are proper, usable and operational, some of them are simpler for the case we need for resource allocation and others do not provide their results by strong mathematical proofs.

In this paper, we assume resources divided between users equally and users with the same resource gain similar utility. In other words, the resources should be divided equally to all users choosing that. In this case, our utility function for choosing each resource is inversely proportional to the number of users in that resource. An early example of proportional sharing is found in research by Waldspurger [39]. Different works have been done on this mechanism after that [40–42]. Therefore, our game model lies in the category of congestion games in which, the utility function is only a function of the number of players using a resource and is player independent. According to the literature of the congestion game, the users are the players compete for some resources. It is shown [43, 44] that for these class of congestion games, in general, there is at least one Nash equilibrium (NE). Also, since a congestion game is a potential game, our game model lies in the category of potential games as well. However, it has been speculated that, in potential games (especially outside of zero-sum games), best response dynamics would rarely converge to mixed equilibria [45–47]. In addition, since we follow a method to find the mix equilibria to define a distribution function for users to choose resources, best response dynamics could not be a proper choice in this case. Moreover, in contrast to fictitious play in a potential game which need a set of initial conditions with Lebesgue measure zero to converge to mix equilibrium [48], we prefer to find the mix equilibrium from all initial values.

In this paper, we propose a model of resource allocation in which, the users choose their resources via a probabilistic bounded rational decision making method. Moreover, we model the interaction among all the users momentarily. Defining the multi-player evolutionary game for this kind of problem lets users have permission to choose their resources in a probabilistic form, act non-cooperatively and compete with all other users momentarily. The asymptotic stable equilibrium points of the replicator dynamic of the evolutionary game also gives the NE point(s) of the game such that no user can benefit by individually deviating from the equilibrium point of the game. Modelling our game, to find the best strategy(i.e. probability function), we need to evaluate the equilibrium point(s) and to discuss on the condition of existence and uniqueness of them. Moreover, to be sure any mutations could not change the strategy, we should prove the stability of the equilibria. According to the above concerns, the major contribution of this paper can be summarised as follows:

- We have proposed a multi-player evolutionary game for a two choice resource allocation problem. To achieve the learning rule of each player, instead of the conventional pairwise matching based evolutionary game model, we use the case where more than two players compete momentarily.
- We derive the multi-player replicator dynamic equation for our resource allocation model.
- We provide the necessary and sufficient conditions for the existence and uniqueness of the mixed equilibrium point of the game.
- We prove the condition in that the game converges to each unique equilibrium point which is asymptotically and evolutionarily stable.

The rest of this paper is organised as follows. Preliminaries, game model is presented in Section 2. Section 3 presents the multi-player replicator dynamics corresponding to the network selection strategy of the users. The results of existence, uniqueness and stability of the equilibrium point and other boundary equilibrium points of the multi-player replicator dynamic are given in this section as well. The simulation results are given in Section 4. Finally, the paper is concluded in Section 5.

## 2 System model

### 2.1 Game theory and evolutionary game: an overview

Game theory is the study of mathematical models for analysing situations in which parties, called players, make decisions. Each player considers the other player's possible decisions, named strategies, in choosing his own strategy. A game defined by a set of players, a set of strategies and a utility(payoff) function that maps players' outcomes to a real number. The utility function affected by the strategies taken by all players. A game could be cooperative or non-cooperative. In both cooperative and non-cooperative games players can communicate but in cooperative games, players make binding agreements, however, in non-cooperative games they independently choose a strategy from a strategy set.

Consider a set of $n$ players denoted by $V = \{1, \dots, n\}$. For $i \in V, u_i: \Omega \to \mathbb{R}$ is the payoff function of player $i$, where $\Omega = \Omega_1 \times \cdots \times \Omega_i \times \cdots \times \Omega_n = \{1, \dots, m\}^n$ is the strategy set of all the players and $\Omega_i$ is the strategy set of player $i$. The game is then denoted by $G(V, \Omega, \{u_i\}_{i \in V})$. Let $x = (x_1, \cdots, x_i, \cdots, x_n)^{\mathrm{T}} \in \Omega$ with $x_i \in \Omega_i$, denotes the strategy of all the players. For simplicity, one can represent $x = (x_i, x_{-i})$ and $\Omega = \Omega_i \times \Omega_{-i}$ where $x_{-i} = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n)^{\mathrm{T}}$ and $\Omega_{-i} = \Omega_1 \times \cdots \times \Omega_{i-1} \times \Omega_{i+1} \times \cdots \times \Omega_n$. The payoff function $u_i$ is a function of all players' strategies (i.e. $x = (x_i, x_{-i})$).

The evolutionary game is known as the application of game theory to evolving populations in biology by extension of Darwinian competition model, however, there are two main groups of practitioners of the concepts of evolutionary game theory. The first game theorists which consider the dynamical system as a model of behavioural evolution in a population where pure strategies (i.e. behaviours) that have higher payoff (i.e. fitness) increase in relative frequency due to their higher reproductive success. In particular, individuals in the population do not make conscious decisions on what strategy to use; rather, these are predetermined by nature. The latter group takes the classical game theory perspective that individuals make rational decisions on their strategy choice [13]. In other words, the probability of choosing possible strategies by the players is the purpose of second group.

In an evolutionary game model, the payoffs for playing a particular strategy depend only on the other strategies employed, not on who is playing them. Such a game with this feature is called a symmetric game. In symmetric game, with $m$ different strategies, $\vec{P} = \{(p_1, \cdots, p_m) \big| p_j \geq 0, \sum p_j = 1\}$ represents a mixed strategy, where $p_j$ is the probability of choosing strategy $j$. Pure strategy $e_j$ is defined as $e_j = (0, \cdots, 0, p_j = 1, 0, \cdots, 0)$. In an evolutionary game, players can update their strategy during the time. This adoption can be modelled by differential equation named *replicator dynamic*. Replicator dynamic is useful to investigate the evolution of strategies towards convergence. For multi-player, two strategies evolutionary game, the expected payoff for each player and the replicator dynamic of a game is defined as follows. The players with $j$th strategy, $j$-strategist, gain the expected payoff $\pi_j$

$$\pi_j = \sum_{k=0}^{n-1} \binom{n-1}{k} p_1^k p_2^{n-1-k} a_j, \tag{1}$$

In this equation, we assume an individual as a main player which can use one of the strategies. $k$ shows the number of co-players of this individual and $n - 1 - k$ shows the number of opponents, also $a_j$ is utility of $j$-strategist players

$$a_j \in \{u_i \big| i \in V, x_i = j\} \tag{2}$$

As a result, a replicator dynamic is defined as

$$\dot{p}_j = p_j(\pi_j - \hat{\pi}) \tag{3}$$

where $\hat{\pi} = \sum_{j=1}^{m} p_j \pi_j$ is the average payoff [15].

### 2.2 Our game model

In our model we have $n$ player and two strategies. Therefore, $V = \{1, \dots, n\}$ and $\Omega = \{1, 2\}^n$. Moreover, to specify our model's utility function, let $U_{m_j}^j$, $j \in \{1, 2\}$ denote the gained payoff for the users with strategy $j$, when $m_j$ users play the strategies $j$ (and $n - m_j$ users play the other single strategy in $\{1, 2\} - \{j\}$). Consider also the number of *ones* in vector $x$ by $m_j$. Then, $m_j$ shows the number of users which choose strategy $j$. We can calculate this value as

$$m_j = (-1)^j(\| x \|_1 - jn) + n$$

Now, according to the definition of $U_{m_j}^j$, $u_i$ is defined as follows:

$$u_i = U_{m_j}^j, j = x_i \tag{4}$$

Moreover, in this model $p_j$ shows the probability of choosing strategy $j$ for each player. Therefore, our model lies on the second interpretation of evolutionary game.

## 3 Network selection

We assume the payoff of a user is inversely proportional to the number of users choosing a specific strategy. This assumption and therefore the following analysis and theories are applicable for a fair resource allocation to a number of users in any system. Medium Access Control (MAC) Protocol is one of the critical issues in the design of wireless sensor networks. As in most wireless networks, collision, which is caused by two nodes sending data at the same time over the same transmission medium, is a great concern in wireless sensor networks (WSNs). To address this problem, a sensor network must employ a MAC protocol to arbitrate access to the shared medium in order to avoid data collision from different nodes and at the same time to fairly and efficiently share the bandwidth resources among multiple sensor nodes. There are many types of MAC protocols, designed for WSN so far. In time-division multiple access (TDMA)-based MAC protocols, time is divided into time-frames and each time-frame is further divided into a fixed number of time-slots [49]. The other promising technologies to improve MAC efficiency is orthogonal frequency division multiple access (OFDMA). With OFDMA technology, the whole channel is divided into several sub-channels, and several subcarriers comprise one sub-channel. Thus, OFDMA enables multiuser channel access and multiuser data transmission since different nodes could use different sub-channels simultaneously [50]. Therefore, TDMA MAC protocols and OFDMA-based MAC protocols with fair subcarrier sharing (e.g. LTE and WiMAX) are examples of proportional sharing in heterogeneous networks [51].

According to our assumption, though the payoff of a user could be written as

$$U_{m_j}^j = \frac{L_j}{m_j}, \tag{5}$$

where $L_j$ shows each RAT's whole recourse. Thus, at first, we find the game's replicator utilising (5).

*Lemma 1:* The multi-player evolutionary game's replicator dynamic with payoff function (5) can be written as

$$\dot{p}_1 = \frac{1}{n} p_1(1 - p_1) f(p_1),$$

where $f(p_1) = \sum_{j=0}^{n-1} p_1^j((-1)^j \binom{n}{j+1} L_1 - L_2)$.

*Proof:* By replacing $a_j = U_{m_j}^j$ to (3), where we have $m_j$ shows the number of $j$-strategist which means the main player and its co-players. Therefore, in $\pi_1$, $m_1 = k + 1$ and in $\pi_2$, $m_2 = n - k - 1 + 1$

$$\begin{aligned}\dot{p}_1 &= p_1 p_2(\pi_1 - \pi_2)\\ &= p_1 p_2 \sum_{k=0}^{n-1} \binom{n-1}{k} p_1^k p_2^{n-1-k} (U_{k+1}^1 - U_{n-k}^2)\end{aligned} \tag{6}$$

Defining $U_* = U_{*+1}^1 - U_{n-*}^2$, replacing $p_2 = 1 - p_1$, and setting $(1 - x)^n = \sum_{i=0}^n \binom{n}{i} (-x)^i$, we can rewrite (6) as

$$\begin{aligned}\dot{p}_1 &= p_1(1 - p_1) \sum_{i=0}^{n-1} \binom{n-1}{i} p_1^i U_i \sum_{j=0}^{n-i-1} \binom{n-i-1}{j} (-p_1)^j\\ &= p_1(1 - p_1) \sum_{i=0}^{n-1} \binom{n-1}{i} U_i \sum_{j=i}^{n-1} \binom{n-i-1}{j-i} (-1)^{j-i} p_1^j\\ &= p_1(1 - p_1) \sum_{j=0}^{n-1} \sum_{i=0}^{j} \binom{n-1}{i} \binom{n-i-1}{j-i} (-1)^{j-i} U_i p_1^j\\ &= p_1(1 - p_1) \sum_{j=0}^{n-1} p_1^j \left( \binom{n-1}{j} \sum_{i=0}^{j} \binom{j}{i} (-1)^{j-i} U_i \right)\end{aligned} \tag{7}$$

Now, by replacing the payoff function from (5), the coefficients of polynomial equation in (7) can be written as

$$\begin{aligned}I_j &= \binom{n-1}{j} \sum_{i=0}^{j} \binom{j}{i} (-1)^{j-i} U_i\\ &= \binom{n-1}{j} \sum_{i=0}^{j} \binom{j}{i} (-1)^{j-i} \left( \frac{L_1}{i+1} - \frac{L_2}{n-i} \right)\\ &= I_{1j} - I_{2j}\end{aligned} \tag{8}$$

We know that

$$\frac{\binom{y}{x}}{x+1} = \frac{\binom{y+1}{x+1}}{y+1}$$

and $\sum_{i=0}^{j} (-1)^i \binom{j}{i} = (1 - 1) = 0$. Therefore, we have

$$\begin{aligned}I_{1j} &= \binom{n-1}{j} \sum_{i=0}^{j} \binom{j}{i} (-1)^{j-i} \frac{L_1}{i+1},\\ &= \frac{L_1}{j+1} \binom{n-1}{j} \sum_{i=0}^{j} \binom{j+1}{i+1} (-1)^{j-i},\\ &= \frac{(-1)^{j-1} L_1}{j+1} \binom{n-1}{j} \left( \sum_{i=0}^{j+1} (-1)^i \binom{j+1}{i} - 1 \right),\\ &= \frac{(-1)^j L_1}{n} \binom{n}{j+1}.\end{aligned} \tag{9}$$

In the same way, we have

$$I_{2j} = \binom{n-1}{j} \sum_{i=0}^{j} \binom{j}{i} (-1)^{j-i} \frac{L_2}{n-i} = \frac{L_2}{n}. \tag{10}$$

As a result, the multi-player replicator dynamic with payoff function (5) becomes

$$\begin{aligned}\dot{p}_1 &= \frac{1}{n} p_1(1 - p_1) \sum_{j=0}^{n-1} p_1^j \left( (-1)^j \binom{n}{j+1} L_1 - L_2 \right),\\ &= \frac{1}{n} p_1(1 - p_1) f(p_1).\end{aligned} \tag{11}$$

□

The replicator dynamic (11) has at least two equilibrium points which are $p_1 = 0$ and $p_1 = 1$. In addition, if $f(p_1)$ has any roots in $[0, 1]$, we can also have some mixed NE points. The Vincent's test is a method which gives the number of roots of a polynomial equation in a specified interval [52] In what follows, we show that $f(p_1)$ is monotonic in $[0, 1]$.

*Vincent's test:* If $a \geq 0$ and $b > 0$, then $Q_{ab}(f)$ the number of real roots in the open interval $(a, b)$, multiplicities counted, of the polynomial $f(p_1) \in \mathbb{R}[p_1]$ is bounded above by the number of sign variations [53] denoted by

$$var_{ab}(f) = var\left(\varphi(p_1)\right) = var\left((1 + p_1)^{deg(f(p_1))} f\left(\frac{a + bp_1}{1 + p_1}\right)\right)$$

and we have $var_{ab}(f) = var_{ba}(f) \geq Q_{ab}(f)$.

*Proof:* See [52]. □

*Lemma 2:* $f(p_1) = \sum_{j=0}^{n-1} p_1^j((-1)^j\binom{n}{j+1}L_1 - L_2)$ is monotonically decreasing in [0, 1].

*Proof:* If $\mathrm{d}f(p_1)/\mathrm{d}p_1$ does not have any roots in $[0, 1]$ and is negative in an arbitrary point on [0, 1], then $f(p_1)$ is negative in the entire interval and as a result, $f(p_1)$ becomes monotonically decreasing. We first verify derivative of $f(p_1)$ as

$$\frac{\mathrm{d}f(p_1)}{\mathrm{d}p_1} = \sum_{j=1}^{n-1} jp_1^{j-1}T_j, \tag{12}$$

where $T_j = ((-1)^j\binom{n}{j+1}L_1 - L_2)$. According to *Vincent's test*, for this problem, we have $a = 0, b = 1$, and the degree of $f(p_1)$ is $n-2$. Also, we have

$$\begin{aligned}\varphi(p_1) &= (1+p_1)^{n-2}\sum_{j=1}^{n-1} j\left(\frac{p_1}{1+p_1}\right)^{j-1}T_j \\ &= \sum_{j=1}^{n-1} jp_1^{j-1}(1+p_1)^{n-j-1}T_j\end{aligned} \tag{13}$$

Now, using

$$(1+p_1)^{n-j-1} = \sum_{i=0}^{n-j-1}\binom{n-j-1}{i}p_1^{(n-j-1)-i}$$

Equation (13) can be rewritten as follows:

$$\begin{aligned}\varphi(p_1) &= \sum_{j=1}^{n-1} jT_j\sum_{k=0}^{n-j-1}\binom{n-j-1}{k}p_1^{n-k-2} \\ &= \sum_{m=0}^{n-2}(n-m-1)T_{n-m-1}\sum_{k=0}^{m}\binom{m}{k}p_1^{n-k-2} \\ &= \sum_{k=0}^{n-2}p_1^{n-k-2}\sum_{m=k}^{n-2}(n-m-1)T_{n-m-1}\binom{m}{k}.\end{aligned} \tag{14}$$

Accordingly, the coefficients of $\varphi(p_1)$ are

$$J_k = \sum_{m=k}^{n-2}(n-m-1)((-1)^{n-m-1}\binom{n}{m}L_1 - L_2)\binom{m}{k} \tag{15}$$

Now, we get $J_k = J_{1k} + J_{2k}$ and simplify each part separately. Replacing

$$\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k} \quad \text{and}$$
$$\sum_{i=k}^{j}(-1)^i\binom{n}{i-k} = (-1)^j\binom{n-1}{j-k},$$

we have

$$\begin{aligned}J_{1k} &= \sum_{m=k}^{n-2}(n-m-1)(-1)^{n-m-1}\binom{n}{k}\binom{n-k}{m-k}L_1 \\ &= \binom{n}{k}(-1)^{n-1}\sum_{m=k}^{n-2}\sum_{i=m}^{n-2}(-1)^m\binom{n-k}{m-k}L_1 \\ &= \binom{n}{k}(-1)^{n-1}(-1)^{n-2}\binom{n-k-2}{n-k-2}L_1 = -\binom{n}{k}L_1\end{aligned} \tag{16}$$

In the same way, according to the fact that

$$\sum_{i=r}^{n}\binom{i}{r} = \binom{n+1}{r+1},$$

we have

$$\begin{aligned}J_{2k} &= \sum_{m=k}^{n-2}(n-m-1)L_2\binom{m}{k} = \sum_{m=k}^{n-2}\sum_{i=m}^{n-2}L_2\binom{m}{k} \\ &= L_2\sum_{i=0}^{k}\sum_{m=k}^{i}\binom{m}{k} = L_2\sum_{i=0}^{k}\binom{i+1}{k+1} = \binom{n}{k+2}L_2.\end{aligned} \tag{17}$$

Therefore, the coefficients $J_k = -\binom{n}{k}L_1 - \binom{n}{k+2}L_2$ are always negative. Hence, $\mathrm{d}f(p_1)/\mathrm{d}p_1$ does not have any roots in $[0, 1]$ and $f(p_1)$ is monotonic. Moreover

$$\left.\frac{\mathrm{d}f(p_1)}{\mathrm{d}p_1}\right|_{p_1=0} = (-1)\binom{n}{2}L_1 - L_2 < 0 \tag{18}$$

Therefore, $f(p_1)$ is monotonically decreasing in [0, 1]. □

In Theorem 1, we will show that (11) has potentially maximum of one mixed NE point. Moreover, the condition for the existence and uniqueness of the mixed NE point is provided.

*Theorem 1:* The multi-player replicator dynamic (11) has a unique mixed NE point if and only if $(L_1/n) < L_2 < nL_1$.

*Proof:* Assume $(L_1/n) < L_2 < nL_1$. Evaluating $f(p_1)$ for $p_1 = 0, p_1 = 1$, we have

$$\begin{aligned}f(p_1) &= \sum_{j=0}^{n-1}p_1^j\left((-1)^j\binom{n}{j+1}L_1 - L_2\right) \\ f(0) &= \left((-1)^0\binom{n}{1}L_1 - L_2\right) = nL_1 - L_2 > 0 \\ f(1) &= \sum_{j=0}^{n-1}\left((-1)^j\binom{n}{j+1}L_1 - L_2\right) = L_1 - nL_2 < 0\end{aligned} \tag{19}$$

According to *Bolzano's theorem* [54], if $f(a)f(b) < 0, \exists c \in (a, b): f(c) = 0$, there is a $p_1^* \in (0, 1)$ such that $f(p_1^*) = 0$. Thus, the multi-player replicator dynamic has at least one mixed Nash point.

Now, we show $f(p_1)$ does not have any root if the condition in Theorem 1 does not maintain. From Lemma 2, we know that $f(p_1)$ is monotonic in (0, 1). Also, we have

$$\text{if } L_2 > nL_1: f(0), f(1) < 0 \Rightarrow \forall p_1 \in (0, 1): f(p_1) < 0$$

In the same way

$$\text{if } L_2 < \frac{L_1}{n}: f(0), f(1) > 0 \Rightarrow \forall p_1 \in (0, 1): f(p_1) > 0$$

Finally, since $f(p_1)$ is monotonic, there is a unique $p_1^* \in (0, 1)$ such that $f(p_1^*) = 0$. □

In Theorem 1, if $n$ is large enough, the authorised region for RATs payoff functions' coefficient almost cover the real values without attention to other RAT payoff functions' coefficient. According to Theorem 1, under the proposed assumption, we have two boundary points and a mixed NE. Otherwise, we only have boundary points. We check the asymptotically stability of equilibrium points in Theorem 2.

*Theorem 2:* The boundary and mixed NE points of (11) are globally asymptotically stable, given the following conditions:

$$
p_{\text{stable}} = \begin{cases} 0, & L_2 < \dfrac{L_1}{n} \\ p_1^*, & \dfrac{L_1}{n} < L_2 < nL_1 \\ 1, & L_2 > nL_1 \end{cases}
$$

where $p_1^*$ is the root of $f(p_1)$ in (11).

*Proof:* To analyse the stability of $p_1^*$, we utilise the Lyapunov function $V = \frac{1}{2}(p_1 - p_{\text{stable}})^2$. In this case, we have

$$
\dot{V} = (p_1 - p_{\text{stable}})\dot{p} = \frac{1}{n}p_1(1 - p_1)(p_1 - p_{\text{stable}})f(p_1) \tag{20}
$$

If $\frac{L_1}{n} < L_2 < nL_1$, we have a unique mixed Nash according to Theorem 1. In this case, we know for $p_{\text{stable}} = p_1^*: f(p_1^*) = 0$ and also in Lemma 2 we showed that $f(p_1)$ is monotonically decreasing which means

$$
\begin{cases} f(p_1) > 0, & p_1 < p_1^* \\ f(p_1) < 0, & p_1 > p_1^* \end{cases} \tag{21}
$$

Therefore, in this case, according to (20), we have

$$
\forall p_1 \in (0, 1), \quad p_1 \neq p_1^*: \dot{V} < 0 \tag{22}
$$

Hence, $p_1^*$ is globally asymptotically stable in $(0, 1)$. Otherwise, as we discussed before, if $L_2 > nL_1$, for all $p_1 \in (0, 1): f(p_1) < 0$ and for $p_{\text{stable}} = 0: (p_1 - p_{\text{stable}}) > 0$ and therefore, $\dot{V} < 0$. In the same way, if $L_2 < L_1/n$, $\forall p_1 \in (0, 1): f(p_1) > 0$ and $p_{\text{stable}} = 1: (p_1 - p_{\text{stable}}) < 0$.

Therefore, in all three cases, $\forall p_1 \in (0, 1), p_1 \neq p_1^*: \dot{V} < 0$ and $p_{\text{stable}}$ is globally stable in its feasible region $(0, 1)$.□

Theorem 2 shows that if one of the RATs has $n$ times more resources than the other, all the users intend to connect to that RAT. Otherwise, the users choose the RATs using the probabilities equal to the mixed NE point of the game.

According to [13], a NE strategy may not be an evolutionarily stable strategy (ESS). In the following theorem, we show that the stable mixed NE point of the game is actually ESS. This means if all the players choose their strategies base on $\tilde{P}$, then this strategy cannot invade if small group play in various strategy which named mutation [15].

*Theorem 3:* The strategy $\tilde{P} = (\tilde{p}_1,\ \tilde{p}_2 = 1 - \tilde{p}_1)^{\mathrm{T}}$ is an ESS in the following conditions:

$$
\tilde{p}_1 = \begin{cases} 0, & nL_1 \leq L_2, n \neq 1 \\ p_1^*, & \dfrac{L_1}{n} < L_2 < nL_1 \\ 1, & \dfrac{L_1}{n} \geq L_2, n \neq 1 \end{cases}
$$

*Proof:* In evolutionary game theory, strategy $\vec{\tilde{P}}$ is said to be ESS if

$$
\begin{cases} \sum_{j=1}^{m} \tilde{p}_j \tilde{\pi}_j > \sum_{j=1}^{m} p_j \tilde{\pi}_j \qquad \text{or} \\ \sum_{j=1}^{m} \tilde{p}_j \tilde{\pi}_j = \sum_{j=1}^{m} p_j \tilde{\pi}_j,\ \sum_{j=1}^{m} \tilde{p}_j \pi_j > \sum_{j=1}^{m} p_j \pi_j \end{cases} \tag{23}
$$

where $p_j \in P$ and $\tilde{\pi}_j$ is a expected payoff of $j$-strategist with strategy $\tilde{P}$. To check (23), we first compute the first expression. Replacing $\tilde{p}_2 = 1 - \tilde{p}_1$ and simplify summation, we have

$$
\begin{aligned} &\sum_{j=1}^{m} (\tilde{p}_j - p_j)\tilde{\pi}_j \\ &= (\tilde{p}_1 - p_1)\sum_{k_1=0}^{n-1} \binom{n-1}{k_1} \tilde{p}_1^{\,k_1} \tilde{p}_2^{\,n-k_1-1} (U_{k_1+1}^1 - U_{n-k_1}^2) \end{aligned} \tag{24}
$$

According to (6)–(11), we have

$$
\sum_{k_1=0}^{n-1} \binom{n-1}{k_1} \tilde{p}_1^{\,k_1} \tilde{p}_2^{\,n-k_1-1} (U_{k_1+1}^1 - U_{n-k_1}^2) = \frac{1}{n} f(\tilde{p}_1) \tag{25}
$$

Therefore

$$
\sum_{j=1}^{m} (\tilde{p}_j - p_j)\tilde{\pi}_j = \frac{1}{n}(\tilde{p}_1 - p_1) f(\tilde{p}_1) \tag{26}
$$

First, we check the mixed Nash point $\tilde{P} = P^* = (p_1^*,\ p_2^* = 1 - p_1^*)^{\mathrm{T}}$ for ESS. We know under the conditions of Theorem 1 that $f(p_1^*) = 0$. Therefore

$$
\sum_{j=1}^{m} (p_j^* - p_j)\pi_j^* = 0 \Rightarrow \sum_{j=1}^{m} p_j^* \pi_j^* = \sum_{j=1}^{m} p_j \pi_j^* \tag{27}
$$

Hence, the second condition should be checked for ESS. Same as (26), we can write

$$
\sum_{j=1}^{m} (p_j^* - p_j)\pi_j = \frac{1}{n}(p_1^* - p_1) f(p_1) \tag{28}
$$

Now, from (21) in Lemma 2 and (28), we get

$$
\forall \vec{P} \in [0, 1]^2, \quad \vec{P} \neq \vec{P}^*: \sum_{j=1}^{m} (p_j^* - p_j)\pi_j > 0 \tag{29}
$$

Therefore, $\vec{P}^*$ is ESS.

In what follows, we check ESS conditions for boundary equilibrium points when $\tilde{p}_1 = 0$ and $\tilde{p}_1 = 1$.

For $\tilde{p}_1 = 0$, from (19), we can rewrite (26) as

$$
\begin{aligned} \sum_{j=1}^{m} (\tilde{p}_j - p_j)\tilde{\pi}_j &= \frac{1}{n}(0 - p_1) f(0) \\ &= \frac{1}{n}(-p_1)(nL_1 - L_2) \end{aligned} \tag{30}
$$

Therefore, $\sum_{j=1}^{m} (\tilde{p}_j - p_j)\tilde{\pi}_j > 0$ if $nL_1 < L_2$. But if $nL_1 = L_2$, then $\sum_{j=1}^{m} (\tilde{p}_j - p_j)\tilde{\pi}_j = 0$, and we should check second condition in this case. From (28) we have

$$
\begin{aligned} \sum_{j=1}^{m} (\tilde{p}_j - p_j)\pi_j &= \frac{1}{n}(-p_1) f(p_1) \\ &= \frac{1}{n}(-p_1)\sum_{j=0}^{n-1} p_1^j \left((-1)^j \binom{n}{j+1} L_1 - nL_1\right) \end{aligned} \tag{31}
$$

For $n \neq 1$, we have $\binom{n}{j+1} < n$, and therefore $f(0) < 0$. As a result, $\sum_{j=1}^{m} (\tilde{p}_j - p_j)\pi_j > 0$ for $nL_1 \leq L_2, n \neq 1$ and $\tilde{p}_1 = 0$ is ESS.

In the same way, for $\tilde{p}_1 = 1$

$$
\begin{aligned} \sum_{j=1}^{m} (\tilde{p}_j - p_j)\tilde{\pi}_j &= \frac{1}{n}(1 - p_1) f(1) \\ &= \frac{1}{n}(1 - p_1)(L_1 - nL_2) \end{aligned} \tag{32}
$$

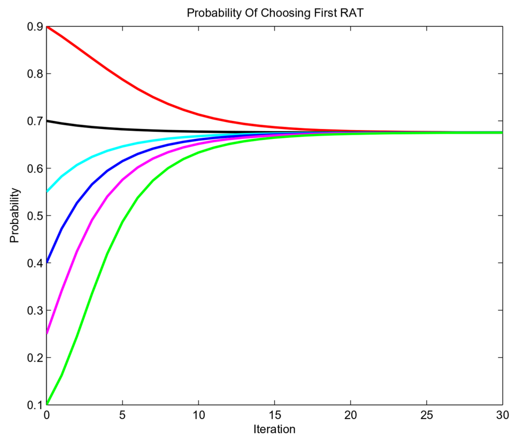


**Fig. 1** *Convergence to the EES*

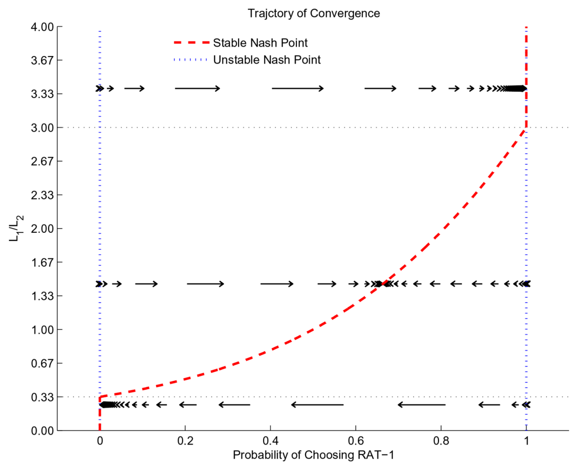


**Fig. 2** *Trajectory of convergence for different proportion of RATs' coefficient*

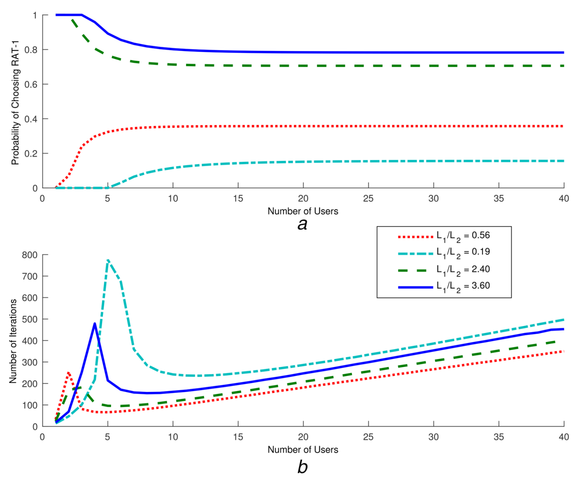


**Fig. 3** *Effect of number of users on*
***(a)*** Final probability of choosing RAT-1, ***(b)*** Number of iterations for convergence

Therefore, $\sum_{j=1}^{m}(\tilde{p}_j - p_j)\tilde{\pi}_j > 0$ if $L_1 > nL_2$. Checking the second condition for $L_1 = nL_2$, from Lemma 2 we know $f(.)$ is decreasing and also from (19) for $L_1 = nL_2$ we have

$$\begin{aligned} f(0) = nL_1 - L_2 = (n^2 - 1)L_2 > 0, n \neq 1 \\ f(1) = L_1 - nL_2 = 0 \end{aligned} \quad (33)$$

Based on these $\forall p_1 \in (0, 1)\colon f(p_1) > 0$ and second condition is met. Therefore, $\tilde{p}_1 = 1$ is ESS if $L_1 \geq nL_2, n \neq 1$. We can conclude $\tilde{P}$ is ESS under the condition of the theorem. □

## 4 Performance evaluation

In this study, we consider a heterogeneous wireless network with two access technologies with throughput functions defined in (5). The effect of parameters of the model and the number of users on the equilibrium point and convergence rate is evaluated.

Fig. 1 shows the convergence of the proposed method for different initial values in the game with four players and $L_1/L_2 = 1.66$. As expected from global stability of the method, despite of the initial value, the game converges to the same point which is NE (i.e. ESS point)

The trajectory of convergence is shown in Fig. 2. The stable Nash point is shown by crossed line and unstable ones by a dotted line. In this simulation, the number of users is 3 and as expected, the system has a mixed Nash point from 1/3 to 3. As shown in this figure, by moving away from equilibrium points, the speed of convergence increases.

Next, we evaluate the convergence rate of the algorithm. Since finding the proper RAT for each user depends on other users' selections, an increment in the number of users extends the time users need to learn the best selection. Also in this selection, our final probability depends on other users and specifically on the number of other users. In Fig. 3, the final probability and the average time of convergence for the different number of users and different RATs' coefficients are shown. In Fig. 3*b*, we have peaked for the number of users near the border of the feasible region of mixed Nash point. As an example for $L_1/L_2 = 3.6$ the peak occurs when there are the three or four users in the game. Otherwise, in all other situation, the number of iterations increases when the number of users moves up.

In the system with 10 players and $L_1/L_2 = 0.5$, the reward that a user gains in any iteration and the average of total profit is shown in Fig. 4. Since this method has a random attribute, we average user's reward in 50 repetitions of the game in the fixed situation. As expected, the profit of user increases as it goes to its equilibrium point and stays almost fixed after convergence. The user almost reaches the maximum of its reward at the 50th iteration in our multi-player evolutionary game.

To compare our results with other methods, we check accumulative utility which each user gains in the process for three different methods. The first method, deterministic, is applied from [9]. The system model used in the mentioned paper is similar to ours. In the deterministic method introduced in that paper, the users change their resource (pure strategy) till they reach to stable NE point and stay on that resource till there is not any change in the environment. The second method, probabilistic, is our method in this paper which user start from initial probability function to choose their resources and dynamically correct their probability function until the stable point. This stable point (probability function) comes from the mixed equilibrium point of the game. In the last method, random, the users act like the probabilistic method but the random probability function is fixed during the process. In this simulation $L1/L2 = 0.54$ and $n = 10$. Fig. 5 shows the result of this comparison.

Each bar shows the distribution of the amount of resource given to each user for different methods in the period of time. As our expectation, we see in the deterministic method, the first three users gain more than others. In probabilistic and random methods, the users' incomes are nearly similar. If we compare these two methods, the difference between the minimum and maximum gain in the probabilistic method is 0.14 and in the random method, it is 0.38. Although these amounts might change according to the probabilistic base of these methods, the users' utilities always have more similarity in the probabilistic method. The definition of NE described the result of this event.

Moreover, in the figure, we can see in the probabilistic and random methods, the sum of probabilities is not 100%. In other words, all the available resources are not used by users. This event happens because it is possible that one of the resources remain empty in some time slots. This occasion might have happened more when the number of users is less or the proportion of two resources goes further from one (the resources are not similar).

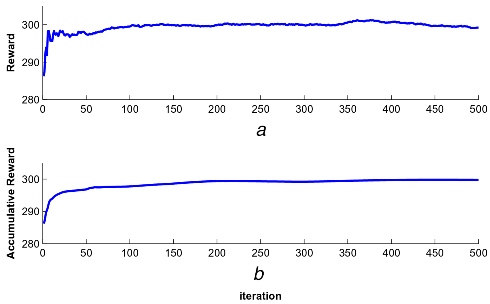


**Fig. 4** *Reward of a user over time*
***(a)*** Mean of user's reward, ***(b)*** Mean of user's accumulative reward

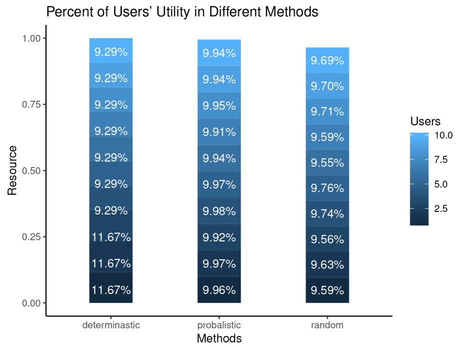


**Fig. 5** *Percent of the users' utility in different methods*

## 5 Conclusion

In this paper, we showed that the multi-player evolutionary game with fair resource allocation has at most one mixed strategy equilibrium point and the necessary and sufficient conditions for existence and uniqueness of such a strategy was provided. In addition, the conditions for global asymptotic stability and evolutionarily stability of boundary equilibrium points as well as mixed NE point of the game were given. As an application of our work, we considered a heterogeneous network where users decide on a probability to select between two access technologies to connect. Based on the proposed multi-player evolutionary game, network selection in heterogeneous networks is studied for a class of utility function. The sensitivity of the equilibrium point and the convergence rate were analysed via simulation results with respect to the parameters of the model and the number of users. The convergence for different initial conditions and different RATs' property are validated as well.

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