Applied Mathematics and Computation 411 (2021) 126464

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Linearization error in synchronization of Kuramoto oscillators

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ARTICLE INFO

Article history:
Received 22 October 2020
Accepted 16 June 2021

Keywords:
Networks
Social networks
Synchronization
Kuramoto model
Linearization
Error estimation

ABSTRACT

Synchronization among a set of networked nodes has attracted much attention in different fields. This paper thoroughly investigates linear formulation of the Kuramoto model, with and without frustration, for an arbitrarily weighted undirected network where all nodes may have different intrinsic frequencies. We develop a mathematical framework to estimate errors of the linear approximation for globally and locally coupled networks. We mathematically prove that the eigenvector corresponding to the largest eigenvalue of the network's Laplacian matrix is enough for examining synchrony alignment and that the functionality of this vector depends on the corresponding eigenvalue. Moreover, we prove that if a globally coupled network with frustration has perfect phase synchronization when its coupling strength tends to infinity, it is a regular network. Finally, the effect of correlation between frustration values and degrees (or frequencies) on the synchronizability of the network is investigated.

1. Introduction

Synchronization as a collective behavior in large ensembles of coupled phase oscillators is encountered in various fields including biological systems [1–3], neuroscience [4,5], computer science [6], physics [7,8], engineering [9,10], social sciences [11,12] and economy [13,14]. From its first modern representations in 1920s [15] to now, much work has been done and reported, considering different aspects, formulations and applications of synchronization. The importance and wide-spread usage of synchronization have kept it a dynamic research area.

In recent years, the extensive use of fast computers and new techniques of simulations have opened a new horizon to the subject. In the light of new visualization techniques, many new statements were added to the synchronization theory, making it richer and appealing (for instance see [16–25]).

The effect of network structure on synchronization has been widely studied (see [5,16,20,23,24,26–36]). In [37], the authors considered the effect of dynamical and structural properties in a network of coupled oscillators on the macroscopic synchronization properties of a system. They investigated optimal intrinsic frequencies for a given network and an optimal network structure for a given set of intrinsic frequencies to optimize synchronization. They introduced Synchrony Alignment Function (SAF) that encodes the interplay between network structure and intrinsic frequencies. The framework of synchrony

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https://doi.org/10.1016/j.amc.2021.126464
0096-3003/

alignment function under the strong coupling regime is used in [38–41], to investigate the relevant problems about synchronization concept. Also, the arrangement problems of network topologies and intrinsic frequencies to optimize network synchronization are investigated by using SAF in undirected [37] and directed networks [42].

Undoubtedly, in a field with such a wide range of applications, the main-stream research path is heavily influenced by its users, sometimes reporting results with lack of mathematical rigor and conclusions solely based on the evidences that come from visualizations and simulations for special scenarios. On the other side, there are formal results that use superfluous assumptions, some even being self-cyclic.

In this paper, we examine the conditions under which the linearization of Kuramoto model and Sakaguchi–Kuramoto model are valid for arbitrarily weighted undirected networks, and provide a framework for considering and estimating the approximation error. Furthermore, we analyse phase-frustrated systems and study the node degree role in synchronization.

This paper is organized as follows. First, we give an overview of phase model of synchronization. Section 3 examines linear coupling function as interaction among oscillators in a coupling network. Section 4 analyses the effect of linearization of coupling functions in obtaining Kuramoto order parameter for weighted Kuramoto dynamics. Section 5 inspects the consequences of linearization in the presence of phase frustration and it is mathematically shown that if the network is not regular, a perfect phase synchronization is impossible for a globally coupled network with frustration even if the coupling strength tends to infinity. Finally, Section 6 presents our conclusions.

2. Preliminaries

In this section, we briefly give some definitions and notations and prove some propositions which will be used later.

2.1. The general phase model of synchronization

Following [37], the dynamics of N coupled phase oscillators that interact through a coupling network can be modelled by

$$\frac{d\theta_i}{dt} = \omega_i + \kappa \sum_{j=1}^{N} A_{ij} H\big(\theta_j - \theta_i\big), \tag{1}$$

where $\theta_i(t)$ is the phase of oscillator i at time t for $i = 1, \ldots, N$, ω_i is the intrinsic frequency of oscillator i, $[A_{ij}]$ is the adjacency matrix of the network, $\kappa > 0$ is the coupling strength, and H is the coupling function, a 2π-periodic function that is differentiable at 0 and governs interactions among all adjacent nodes of the network. In the classical models, Kuramoto [43] and Sakaguchi–Kuramoto [44], this periodic function is $H(\theta) = \sin\theta$ and $H(\theta) = \sin(\theta - \alpha)$, respectively, where $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$ is the phase-lag parameter. A classical approach to deal with the dynamics indicated in Eq. (1) is linearizing it under the assumption that the difference of $\theta_j - \theta_i$ is small enough for all $1 \le i < j \le N$. By linearization, it is obtained that

$$\frac{d\theta_i}{dt} = \omega_i + \kappa H(0) d_i - \kappa H'(0) \sum_{j=1}^{N} L_{ij}\theta_j, \tag{2}$$

where d_i is the degree of node i and $[L_{ij}]$ is Laplacian matrix of the network (see Appendix B).

2.2. The Kuramoto model

Our representation of the Kuramoto dynamics is a generalization of that in Eq. (1), in the sense that we consider a weighted network with (possibly) different coupling strengths instead of a fixed coupling strength for all adjacent nodes, and a family of coupling functions instead of just one coupling function. Here, we consider the equation system

$$\frac{d\theta_i}{dt} = \omega_i + \sum_{j=1}^{N} \kappa_{ij} H_{ij}(\theta_j - \theta_i) \tag{3}$$

for all $1 \le i \le N$, that models the evolution of phase $\theta_i(t)$ for oscillator i in the dynamic interaction among N coupled oscillators with different intrinsic frequencies ω_i, and for all $1 \le i < j \le N$ the positive number κ_{ij} represents the symmetric coupling strength between oscillators i and j if there is a link between them, and $\kappa_{ij} = 0$ otherwise. Also each coupling function $H_{ij}(\theta)$ is a 2π-periodic function such that $H_{ij}(0) = 0$ and has nonzero first derivative at zero. To our best knowledge, although this generalization has been indicated in several articles, it is probed only in its simple form with a global coupling strength. We call this system a *locally coupled network* and the system presented by Eq. (1) is called a *globally coupled network*.

Suppose that $K = [\kappa_{ij}]$ is the weighted adjacency matrix of the connected network of oscillators represented by a weighted graph of N nodes where the i-th oscillator is placed on vertex i of the network. Let L be the Laplacian matrix of the network such that $L_{ij} = \delta_{ij} d_i - \kappa_{ij}$ for each $1 \le i, j \le N$ where $d_i = \sum_{j=1}^{N} \kappa_{ij}$ is the sum of weights of edges that are connected to vertex v_i, and δ_{ij} is 1 if $i = j$ and 0 otherwise. As we assume that the network is connected, the eigenvalues

of the Laplacian matrix L are non-negative real numbers and we denote them by $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_N$ (see Theorem 3 in Appendix A). Let $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_N$ be the orthonormal eigenvalues of L corresponding to $\lambda_1, \ldots, \lambda_N$ respectively; i. e. $L\boldsymbol{v}_i = \lambda_i \boldsymbol{v}_i$ for $i = 1, \ldots, N$. Note that $\boldsymbol{v}_1^T = \frac{1}{\sqrt{N}}\mathbf{1}$ where $\mathbf{1} = [1 \ \ldots \ 1]^T$.

Assume $L^\dagger = \sum_{j=2}^{N} \frac{1}{\lambda_i} \boldsymbol{v}_j \boldsymbol{v}_j^T$. Thus $L^\dagger \boldsymbol{v}_i = \frac{1}{\lambda_i}\boldsymbol{v}_i$ for each $i > 1$ and $L^\dagger \boldsymbol{v}_1 = 0$. Note that $L^\dagger$ is the inverse of L on $\text{span}(\{\boldsymbol{v}_2, \ldots, \boldsymbol{v}_N\})$ where $\text{span}(V)$ is the vector space generated by V, i.e. the set of all linear combinations of all vectors in V.

Finally, if a vector $\boldsymbol{\theta}(t) = [\theta_1(t) \ \ldots \ \theta_N(t)]^T$ satisfies the system in Eq. (3) with $\boldsymbol{\theta}(0) = \boldsymbol{\theta}_0$ then we say "$\boldsymbol{\theta}(t)$ satisfies the system determined by $\langle \boldsymbol{\omega}, \boldsymbol{\theta}_0, K, H\rangle$" where $\boldsymbol{\omega} = [\omega_1 \ \ldots \ \omega_N]^T$ and H is a two dimensional array formed of H_{ij}'s.

2.3. The order parameter

The degree of phase synchronization of an ensemble of oscillators in the network is measured by the Kuramoto order parameter which is defined as

$$r_{\langle \boldsymbol{\omega}, \boldsymbol{\theta}_0, K, H\rangle}(t) = \frac{1}{N}\left| \sum_{j=1}^{N} e^{i\theta_j(t)} \right|,$$

where $\boldsymbol{\theta}$ satisfies the system determined by $\langle \boldsymbol{\omega}, \boldsymbol{\theta}_0, K, H\rangle$. The parameter $r_{\langle \boldsymbol{\omega}, \boldsymbol{\theta}_0, K, H\rangle}(t)$ lies in $[0, 1]$ and measures the coherence of the oscillator population. Setting $R(t) = (r_{\langle \boldsymbol{\omega}, \boldsymbol{\theta}_0, K, H\rangle}(t))^2$, we have

$$R(t) = \frac{1}{N^2}\left| \sum_{j=1}^{N} e^{i\theta_j(t)} \right|^2 = \frac{1}{N^2}\sum_{i,j} \cos(\theta_i - \theta_j). \tag{4}$$

Detailed proofs for Eq. (4) and Propositions 1–3 can be found in Appendix B.

Proposition 1. *If $\boldsymbol{\theta}$ satisfies the system $\langle \boldsymbol{\omega}, \boldsymbol{\theta}_0, K, H\rangle$ and $|\theta_i(t) - \theta_j(t)| < \varepsilon \leq 1$ for a positive ε, all $1 \leq i, j \leq N$ and all $t > \tau$ for some $\tau \in \mathbb{R}$, then*

$$R(t) \geq \sqrt{1 - \varepsilon^2}. \tag{5}$$

Proposition 2. *If $\boldsymbol{\theta}$ satisfies the system $\langle \boldsymbol{\omega}, \boldsymbol{\theta}_0, K, H\rangle$, then*

$$R(t) \geq 1 - \frac{1}{N}\|\boldsymbol{\theta}(t) - \overline{\theta}(t)\mathbf{1}\|^2, \tag{6}$$

where $\overline{\theta}(t) = \frac{\theta_1(t) + \cdots + \theta_N(t)}{N}$.

In the rest of this section, we examine the impact of changes for average and standard deviation of frequency vectors on the order parameter.

Proposition 3. *If $\boldsymbol{\theta}$ satisfies the system $\langle \boldsymbol{\omega}, \boldsymbol{\theta}_0, K, H\rangle$ then $\boldsymbol{\varphi}(t) = \boldsymbol{\theta}(t) + ct\mathbf{1}$ satisfies the system $\langle \boldsymbol{\omega} + c\mathbf{1}, \boldsymbol{\theta}_0, K, H\rangle$ and $\boldsymbol{\psi}(t) = \boldsymbol{\theta}(\frac{1}{c}t)$ satisfies the system $\langle \frac{1}{c}\boldsymbol{\omega}, \boldsymbol{\theta}_0, \frac{1}{c}K, H\rangle$ for any real number $c > 0$.*

Proposition 3 implies that the value of the order parameter remains as same for both $\boldsymbol{\theta}$ and $\boldsymbol{\varphi}$ since $\varphi_j - \varphi_i = \theta_j - \theta_i$, which shows that the average of ω_i's has no effect on the order parameter. Hence, without loss of generality, we can assume that the average of frequency vector is zero. Moreover, it shows that one may rescale ω freely to get an arbitrary standard deviation, at the cost of rescaling time and coupling strengths. We summarize the point as follows:

Corollary 1. *If $\boldsymbol{\theta}$ satisfies the system $\langle \boldsymbol{\omega}, \boldsymbol{\theta}_0, K, H\rangle$ then for any real number $c > 0$, we have*

$$r_{\langle \boldsymbol{\omega}, \boldsymbol{\theta}_0, K, H\rangle}(t) = r_{\langle \boldsymbol{\omega} + c\mathbf{1}, \boldsymbol{\theta}_0, K, H\rangle}(t),$$
$$r_{\langle \boldsymbol{\omega}, \boldsymbol{\theta}_0, K, H\rangle}(t) = r_{\langle \frac{1}{c}\boldsymbol{\omega}, \boldsymbol{\theta}_0, \frac{1}{c}K, H\rangle}(t).$$

Recall that if $\bar{\omega} = 0$ where $\bar{\omega}$ is the average of $\boldsymbol{\omega}$, then the standard deviation of $\boldsymbol{\omega}$, $\sigma(\boldsymbol{\omega})$, is equal to $\frac{1}{\sqrt{N}}\|\boldsymbol{\omega}\|$; so if $\bar{\omega} = 0$, then any constraint on either standard deviation or norm of $\boldsymbol{\omega}$ corresponds naturally to a constraint on the other one.

3. Weighted connected networks with linear coupling function

Consider the linear coupling function $H_{ij}(\theta) = \theta$ for each $1 \leq i < j \leq N$. Thus, θ_i's satisfy

$$\frac{d\theta_i}{dt} = \omega_i + \sum_{j=1}^{N} \kappa_{ij}(\theta_j - \theta_i) \tag{7}$$

Note that, this is essentially the same as Eq. (2) for $H_{ij}(\theta) = \theta$, but we may have different κ's. We can rewrite Eq. (7) as

$$\frac{d\boldsymbol{\theta}}{dt} = \boldsymbol{\omega} - L\boldsymbol{\theta}. \tag{8}$$

Using the inverse Laplace transform, we obtain (see Appendix B)

$$\begin{aligned}\boldsymbol{\theta}(t) &= t\boldsymbol{v}_1\boldsymbol{v}_1^T\boldsymbol{\omega} + \boldsymbol{v}_1\boldsymbol{v}_1^T\boldsymbol{\theta}(0) + \sum_{j=2}^{N}\left(\boldsymbol{v}_j\boldsymbol{v}_j^T\left(\frac{-1}{\lambda_j}\boldsymbol{\omega} + \boldsymbol{\theta}(0)\right)e^{-\lambda_j t} + \frac{1}{\lambda_j}\boldsymbol{v}_j\boldsymbol{v}_j^T\boldsymbol{\omega}\right)\\ &= \left(\bar{\omega}t + \bar{\theta}(0)\right)\mathbf{1} + \sum_{j=2}^{N}\left(\boldsymbol{v}_j\boldsymbol{v}_j^T\left(\frac{-1}{\lambda_j}\boldsymbol{\omega} + \boldsymbol{\theta}(0)\right)e^{-\lambda_j t} + \frac{1}{\lambda_j}\boldsymbol{v}_j\boldsymbol{v}_j^T\boldsymbol{\omega}\right).\end{aligned} \tag{9}$$

It can be verified easily that, for a given positive ε, if

$$t > -\frac{1}{\lambda_2}\ln\frac{\lambda_2\varepsilon}{N(\|\boldsymbol{\omega}\| + \lambda_2\|\boldsymbol{\theta}(0)\|)},$$

then

$$\left\|\sum_{j=2}^{N}\boldsymbol{v}_j\boldsymbol{v}_j^T\left(\frac{-1}{\lambda_j}\boldsymbol{\omega} + \boldsymbol{\theta}(0)\right)e^{-\lambda_j t}\right\| \le \varepsilon,$$

which shows that, as $t \to \infty$, there exists an asymptotic solution for the differential equation system (8) given by

$$\boldsymbol{\theta}(t) = \left(\bar{\omega}t + \bar{\theta}(0)\right)\mathbf{1} + \sum_{j=2}^{N}\frac{1}{\lambda_j}\boldsymbol{v}_j\boldsymbol{v}_j^T\boldsymbol{\omega}.$$

So, $\frac{d\boldsymbol{\theta}}{dt} = \bar{\omega}\mathbf{1}$ that shows

$$\frac{d\theta_1}{dt} = \cdots = \frac{d\theta_N}{dt} = \overline{\omega} \tag{10}$$

when $t \to \infty$. As a result, the frequencies of all oscillators are finally identical, which means that the oscillators rotate with the same velocity, best known as phase-lock synchronization [45]. Since $L^\dagger = \sum_{j=2}^{N}\frac{1}{\lambda_j}\boldsymbol{v}_j\boldsymbol{v}_j^T$,we have $L^\dagger\boldsymbol{v}_i = \frac{1}{\lambda_i}\boldsymbol{v}_i$ for $1 < i \le N$ and $L^\dagger\boldsymbol{v}_1 = 0$, and hence we may rewrite $\boldsymbol{\theta}$ as

$$\boldsymbol{\theta}(t) = \left(\bar{\omega}t + \bar{\theta}(0)\right)\mathbf{1} + L^\dagger\boldsymbol{\omega}. \tag{11}$$

Since $\boldsymbol{v}_1^T\boldsymbol{v}_j = 0$ for $2 \le j \le N$, multiplying both sides of Eq. (11) by $\boldsymbol{v}_1^T$ from left gives

$$\bar{\theta}(t) = \bar{\omega}t + \bar{\theta}(0),$$

and, consequently, Eq. (11) can be written as

$$\boldsymbol{\theta} - \bar{\theta}\mathbf{1} = L^\dagger\boldsymbol{\omega}. \tag{12}$$

To give a more explicit relation between differences and $\boldsymbol{\omega}$, let $\boldsymbol{\theta}^* = \boldsymbol{\theta} - \bar{\theta}\mathbf{1}$, and split $\boldsymbol{\omega}$ as $c\boldsymbol{v}_1 + \boldsymbol{v}$ where $\boldsymbol{v} \in \text{span}(\{\boldsymbol{v}_2, \ldots, \boldsymbol{v}_N\})$ and $c \in \mathbb{R}$; thus $L^\dagger\boldsymbol{v} = L^\dagger\boldsymbol{\omega} = \boldsymbol{\theta}^*$ by Eq. (12). Since both $\boldsymbol{v}$ and $\boldsymbol{\theta}^*$ are in $\text{span}(\{\boldsymbol{v}_2, \ldots, \boldsymbol{v}_N\})$, we have $\boldsymbol{v} = L\boldsymbol{\theta}^*$ and hence

$$\boldsymbol{\omega} = k\boldsymbol{v}_1 + L\boldsymbol{\theta}^*. \tag{13}$$

Furthermore, as a result,

$$\|\boldsymbol{\theta} - \bar{\theta}\mathbf{1}\| = \|L^\dagger\boldsymbol{\omega}\|,$$

and hence by Inequality (6) we have

$$R(t) \ge 1 - \frac{1}{N}\|\boldsymbol{\theta}(t) - \overline{\theta}(t)\mathbf{1}\|^2 = 1 - \frac{1}{N}\|L^\dagger\boldsymbol{\omega}\|^2. \tag{14}$$

That is, if we define $Lb(\boldsymbol{\omega})$ by

$$Lb(\boldsymbol{\omega}) = 1 - \frac{1}{N}\|L^\dagger\boldsymbol{\omega}\|^2,$$

then $Lb(\boldsymbol{\omega})$ is a lower bound for $R(t)$ and, hence, $R(t)$ must stay in the closed interval $[Lb(\boldsymbol{\omega}), 1]$. If we choose $\boldsymbol{\omega}$ with respect to an explicit difference vector $\boldsymbol{\theta}^*$ as in Eq. (13), then we have

$$Lb(\boldsymbol{\omega}) = 1 - \frac{1}{N}\|\boldsymbol{\theta}^*\|^2.$$

In the following lemma, we show that vectors $\boldsymbol{v}_2, \ldots, \boldsymbol{v}_{N-1}$ have no essential role in $Lb(\boldsymbol{\omega})$.

Lemma 1. *Let*

$$\begin{cases} V_1 = \text{span}(\{\boldsymbol{v}_1\}), \\ V_{i+1} = V_i + \text{span}(\{\boldsymbol{v}_{N-i+1}\}), \quad 1 \le i < N. \end{cases}$$

For each $\boldsymbol{\omega} \in V_{i+1} \setminus V_i$, there is $\boldsymbol{\omega}' \in V_i$ such that $\|\boldsymbol{\omega}'\| = \|\boldsymbol{\omega}\|$ and $Lb(\boldsymbol{\omega}') \ge Lb(\boldsymbol{\omega})$.

Proof. Suppose that $\boldsymbol{\omega} = \boldsymbol{v} + c\boldsymbol{v}_{N-i+1}$ where $\boldsymbol{v} \in V_i$ and $c \in \mathbb{R}$. Note that $c \neq 0$, since $\boldsymbol{\omega} \notin V_i$. If $\boldsymbol{v}$ is zero then put

$$\boldsymbol{\omega}' = \begin{cases} c\boldsymbol{v}_1 & i = 1, \\ c\boldsymbol{v}_{n-i+2} & i \ge 2. \end{cases}$$

Trivially, $\|\boldsymbol{\omega}'\| = \|\boldsymbol{\omega}\|$ and $Lb(\boldsymbol{\omega}') \ge Lb(\boldsymbol{\omega})$.

If $\boldsymbol{v}$ is not zero then set $\boldsymbol{\omega}' = \frac{\|\boldsymbol{\omega}\|}{\|\boldsymbol{v}\|}\boldsymbol{v}$. If we denote the restriction of $L^\dagger$ to V_i by $L^\dagger_{|V_i}$, then the biggest eigenvalue of $L^{\dagger 2}_{|V_i}$ is $\frac{1}{\lambda^2_{n-i+2}}$ and we have

$$\frac{\boldsymbol{v}^T (L^{\dagger 2})\boldsymbol{v}}{\boldsymbol{v}^T\boldsymbol{v}} \le \frac{1}{\lambda^2_{n-i+2}}$$

by the Rayleigh's principle [46, p.11]). Now, as $\boldsymbol{v} \perp \boldsymbol{v}_{n-i+1}$ and $c \neq 0$, it follows that

$$\begin{aligned} \langle L^\dagger\boldsymbol{\omega}, L^\dagger\boldsymbol{\omega}\rangle &= \langle L^\dagger\boldsymbol{v}, L^\dagger\boldsymbol{v}\rangle + c^2\langle L^\dagger\boldsymbol{v}_{n-i+1}, L^\dagger\boldsymbol{v}_{n-i+1}\rangle \\ &= \langle L^\dagger\boldsymbol{v}, L^\dagger\boldsymbol{v}\rangle + \frac{c^2}{\lambda^2_{n-i+1}} \\ &\ge \langle L^\dagger\boldsymbol{v}, L^\dagger\boldsymbol{v}\rangle + \frac{c^2}{\lambda^2_{n-i+2}} \\ &\ge \langle L^\dagger\boldsymbol{v}, L^\dagger\boldsymbol{v}\rangle + \frac{c^2}{\|\boldsymbol{v}\|^2}\langle L^\dagger\boldsymbol{v}, L^\dagger\boldsymbol{v}\rangle \\ &= \frac{\|\boldsymbol{\omega}\|^2}{\|\boldsymbol{v}\|^2}\langle L^\dagger\boldsymbol{v}, L^\dagger\boldsymbol{v}\rangle \\ &= \langle L^\dagger\boldsymbol{\omega}', L^\dagger\boldsymbol{\omega}'\rangle. \end{aligned}$$

■

Note that when the average of $\boldsymbol{\omega}$ is zero, the above construction gives $\boldsymbol{\omega}'$ with average zero.

Corollary 2. *Under the assumptions of Lemma 1, if $i > 1$ and $\langle\boldsymbol{\omega}, \boldsymbol{v}_1\rangle = 0$ then $\boldsymbol{\omega}'$ can be taken so that $\langle\boldsymbol{\omega}', \boldsymbol{v}_1\rangle = 0$.*

Corollary 3. *For a given positive σ, the least value of $\|L^\dagger\boldsymbol{\omega}\|$ over all vectors $\boldsymbol{\omega}$ with $\sigma(\boldsymbol{\omega}) = \sigma$ is $\frac{\sigma\sqrt{N}}{\lambda_N}$ which can be obtained by $\boldsymbol{\omega} = k\boldsymbol{v}_1 + \frac{\sigma\sqrt{N}}{\lambda_N}\boldsymbol{v}_N$ for any real number k. The same vector $\boldsymbol{\omega}$ gives the largest value of $Lb(\boldsymbol{\omega})$ with the same constraint, which is $1 - \frac{\sigma^2}{\lambda_N^2}$.*

The above corollary shows that the optimum allocation frequency vector with a fixed standard deviation σ for the lower bound of synchronization is $\boldsymbol{\omega} = c\boldsymbol{v}_1 + \frac{\sigma\sqrt{N}}{\lambda_N}\boldsymbol{v}_N$ for any real number c in a given fixed network of oscillators, and consequently, $R(t) \ge 1 - \frac{\sigma^2}{\lambda_N^2}$. This is depicted in Fig. 1, where the curves show the values of the lower bound function ($Lb(\boldsymbol{\omega})$) versus coupling constants for the Barabasi–Albert networks [47] (i.e. scale-free networks with $\gamma = 3$), considering different eigenvectors as the intrinsic frequencies. The results show the fact that the eigenvector corresponding to the largest eigenvalue of the Laplacian matrix is sufficient for aligning the frequencies. Therefore, using other eigenvectors or a linear combination of eigenvectors for the frequency alignment does not lead to a higher synchronizability.

It should be noted that the dynamics we presented in Eq. (7) is not a Kuramoto system in the sense of Eq. (3), as we have no 2π-periodic function H. However, if we take a suitable $\boldsymbol{\omega}$ which gives a sufficiently small value for $\|L^\dagger\boldsymbol{\omega}\|$ (for example $\|L^\dagger\boldsymbol{\omega}\| < 1$), then we have

$$\begin{aligned} |\theta_i - \theta_j| &\le |\theta_i - \overline{\theta}| + |\overline{\theta} - \theta_j| \\ &\le 2\|\boldsymbol{\theta} - \overline{\theta}\mathbf{1}\| \\ &= 2\|L^\dagger\boldsymbol{\omega}\| \\ &\le 2, \end{aligned}$$

and hence we may take H_{ij} as the identity function on $[-\pi, \pi]$ and extend it as a 2π-periodic function to $\mathbb{R}$.

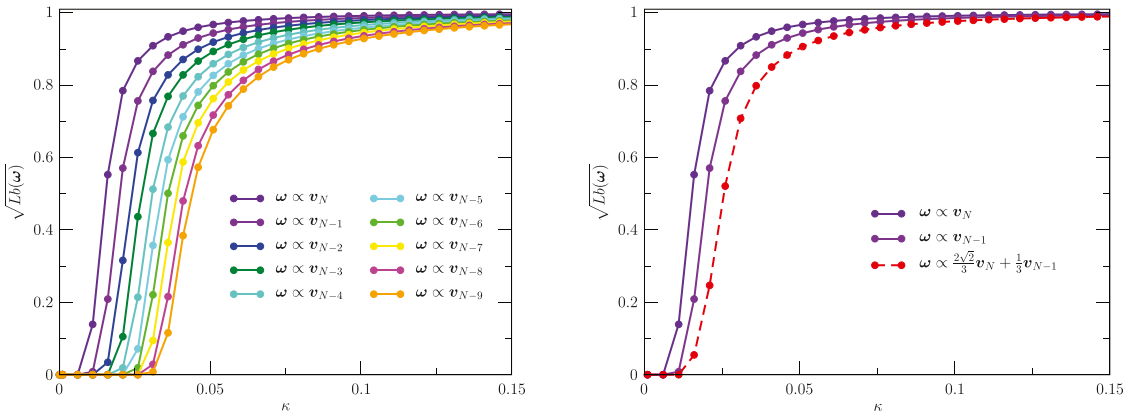

Fig. 1. The eigenvector corresponding to the largest eigenvalue of the Laplacian matrix is sufficient for the frequency alignment. Lower bound values versus the couplings for different frequency alignments; i.e. $\boldsymbol{\omega} = \sqrt{N}\boldsymbol{v}_N, \sqrt{N}\boldsymbol{v}_{N-1}, \ldots, \sqrt{N}\boldsymbol{v}_{N-10}$ and $\boldsymbol{\omega} = \frac{2\sqrt{2N}}{3}\boldsymbol{v}_N + \frac{\sqrt{N}}{3}\boldsymbol{v}_{N-1}$. Each point is averaged over 50 realizations of Barabasi–Albert networks with $N = 1000$ and average degree 4.

4. Weighted Kuramoto dynamics

Suppose that $\boldsymbol{\theta}$ satisfies the system $\langle \boldsymbol{\omega}, \boldsymbol{\theta}_0, K, H\rangle$ where $H_{ij}(\theta) = \sin\theta$ for each $1 \le i < j \le N$. To give a linear approximation for the system (3), we may use Taylor's Theorem for $\sin\theta$ as

$$\sin\theta = \theta - \frac{c^3}{6}$$

for some $c \in \big(-|\theta|, |\theta|\big)$. So if we choose $\boldsymbol{\omega}$ such that $\|L^\dagger\boldsymbol{\omega}\| < \varepsilon$ for some $\varepsilon > 0$, then by Eq. (12), we have $\|\boldsymbol{\theta} - \tilde{\theta}\mathbf{1}\| < \varepsilon$. Consequently, since $|\theta_j - \theta_i| \le 2\varepsilon$, we have

$$\sin(\theta_j - \theta_i) \approx \theta_j - \theta_i$$

with a maximum error of $\frac{4}{3}\varepsilon^3$, which shows that the Kuramoto dynamics can be well-approximated by

$$\frac{d\theta_i}{dt} = \omega_i + \sum_{j=1}^{N} \kappa_{ij}(\theta_j - \theta_i) \tag{15}$$

for each $i = 1, \ldots, N$. For a fixed $\varepsilon > 0$, we say a Kuramoto system $\langle \boldsymbol{\omega}, \boldsymbol{\theta}_0, K, H\rangle$ is ε-*linearizable* if $\|L^\dagger\boldsymbol{\omega}\| \le \varepsilon$. In this situation, by Eq. (6), we obtain

$$R(t) \ge 1 - \frac{\varepsilon^2}{N},$$

which ensures that for a sufficiently small ε, its lower bound of the order parameter, $Lb(\boldsymbol{\omega})$, and consequently its stationary order parameter is near enough to 1 already. Also, if the Kuramoto system $\langle \boldsymbol{\omega}, \boldsymbol{\theta}_0, K, H\rangle$ is ε-linearizable for a sufficiently small ε, then Eqs. (8)–(14) still hold. Thus, all oscillators will be eventually in phase-lock synchronization state.

If we assume $\kappa_{ij} = \kappa > 0$ for $1 \le i < j \le N$, then we have $L = \kappa L_u$ where L_u stands for standard (unweighted) Laplacian matrix of the network. So $L^\dagger = \frac{1}{\kappa}L_u^\dagger$. Assume that $0 = \lambda_1(L_u) < \lambda_2(L_u) \le \ldots \le \lambda_N(L_u)$ are the eigenvalues of L_u. In [37], authors derived the synchrony alignment function

$$J(\boldsymbol{\omega}, L_u) = \frac{1}{N}\sum_{j=2}^{N} \frac{1}{\lambda_j^2(L_u)} \langle \boldsymbol{v}_j, \boldsymbol{\omega}\rangle^2$$

for the Kuramoto dynamics on a network with a fixed coupling strength to measure the Kuramoto order parameter. Since

$$J(\boldsymbol{\omega}, L_u) = \frac{1}{N}\langle L_u^\dagger\boldsymbol{\omega}, L_u^\dagger\boldsymbol{\omega}\rangle = \frac{\kappa^2}{N}\langle L^\dagger\boldsymbol{\omega}, L^\dagger\boldsymbol{\omega}\rangle,$$

the value of the synchrony alignment function $J(\boldsymbol{\omega}, L_u)$ is determined by the linearization error. Moreover, it corresponds to our lower bound function $Lb(\boldsymbol{\omega})$ by

$$\begin{aligned} Lb(\boldsymbol{\omega}) &= 1 - \tfrac{1}{N}\|L^\dagger\boldsymbol{\omega}\|^2 \\ &= 1 - \tfrac{1}{\kappa^2}J(\boldsymbol{\omega}, L_u), \end{aligned}$$

which shows that increasing the coupling strength or decreasing the heterogeneity of the frequencies gives a better lower bound for the order parameter.

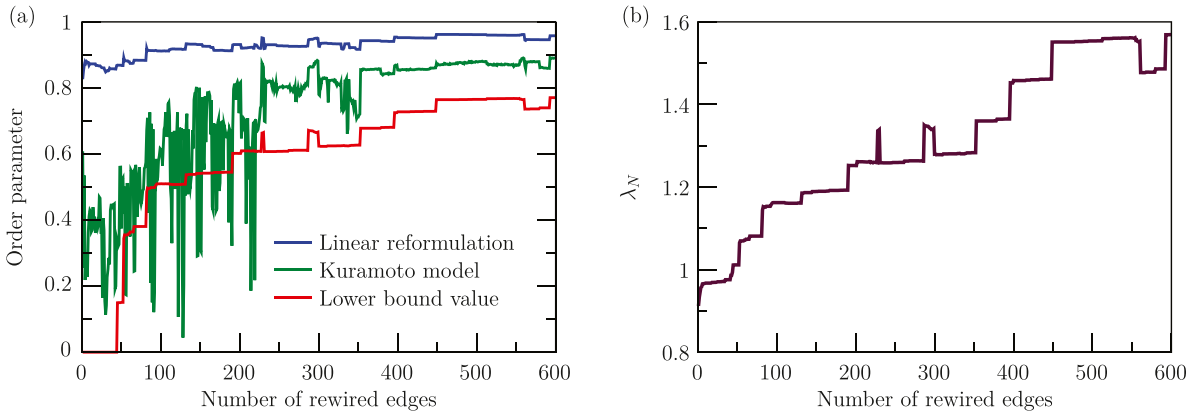

Fig. 2. The linearization error for the Kuramoto model on the small-world networks. (a) The stationary order parameters for the Kuramoto model and its linear reformulation, and the corresponding lower bound values, versus number of rewired edges. (b) The largest eigenvalues of the Laplacian matrices versus the number of the rewired edges. Here the networks have $N = 200$ nodes, average degree 6, and the coupling has been considered as $\kappa = 0.1$.

In [37], the intrinsic frequency vector $\boldsymbol{\omega} = \sqrt{N}\boldsymbol{v}_N$ is examined as a frequency vector to optimize synchronization among frequency vectors whose average and standard deviations are zero and one, respectively. For a fixed coupling strength κ, since

$$L^{\dagger}(\sqrt{N}\boldsymbol{v}_N) = \frac{\sqrt{N}}{\kappa\lambda_N(L_u)}\boldsymbol{v}_N,$$

a linear approximation works well when $\frac{\sqrt{N}}{\kappa\lambda_N(L_u)}$ is small enough; but when it is not so, the Kuramoto system $\langle \boldsymbol{\omega} = \sqrt{N}\boldsymbol{v}_N, \boldsymbol{\theta}_0, \kappa L_u, H\rangle$ is not ε-linearizable for a small enough ε. As an example of such a situation, consider a cycle with N nodes. The eigenvalues of its standard Laplacian matrix are $2 - 2\cos(\frac{2\pi k}{N})$ for $k = 1, \ldots, N$ and hence none of them is greater than 4; so, we have

$$\|L^{\dagger}(\sqrt{N}\boldsymbol{v}_N)\| \geq \frac{\sqrt{N}}{4\kappa}$$

for large enough N and fixed coupling strength κ. For another example, consider grid graphs. By Theorem 4, the biggest eigenvalue of the standard Laplacian matrix is not greater than 8.

More generally, consider a family $\mathcal{F}$ of connected networks with the following two properties:

1. For each $n \in \mathbb{N}$ there is a $G \in \mathcal{F}$ with more than n nodes,
2. There is a natural number k that for each $G \in \mathcal{F}$, there is no node of G with degree higher than k.

We call such a family *uniformly bounded*, and the number k is called a uniform bound on degrees of elements of $\mathcal{F}$. The family of all paths, the family of all cycles, the family of all grids and the family of all k-regular networks (for a fixed k) are uniformly bounded.

Theorem 1. *A uniformly bounded family of networks has a uniform bound on its eigenvalues.*

Proof. Let $\mathcal{F}$ be a uniformly bound family of networks, so there is a natural number k that is a uniform bound on degrees of elements of $\mathcal{F}$. If we take a $G \in \mathcal{F}$, then by Theorem 4, all eigenvalues of the standard Laplacian matrix of G are less than or equal to $2k$.■

For a given uniformly bounded family of networks, the fraction $\frac{\sqrt{N}}{\lambda_N}$ takes arbitrary big values, and hence the Kuramoto system $\langle \sqrt{N}\boldsymbol{v}_N, \boldsymbol{\theta}_0, \kappa L_u, \sin\rangle$ cannot be ε-linearizable for a small enough ε for a fixed strength coupling κ. We now illustrate our results by some numerical examples in which we show that synchrony alignment function cannot provide a robust approximation of the value of Kuramoto order parameter.

To this end, some well-known complex networks are presented that their Kuramoto systems are not ε-linearizable for small enough ε. As a first example, Fig. 2(a) compares the synchronizability of the small-world Watts–Strogatz networks [48] measured by different order parameters: the stationary order parameter for the Kuramoto model, its linear reformulation, and the lower bound function ($\sqrt{Lb(\boldsymbol{\omega})}$). We can see that for the small number of rewired edges, the Kuramoto system is not ε-linearizable, thus Kuramoto order parameter of the linearization of the system $\langle \boldsymbol{\omega}, \boldsymbol{\theta}_0, K, \sin\rangle$ depicted in Fig. 2(a) by blue color has no relation with Kuramoto order parameter of $\langle \boldsymbol{\omega}, \boldsymbol{\theta}_0, K, \sin\rangle$; so, one cannot choose $\sqrt{N}\boldsymbol{v}_N$ as an optimal allocation in these cases. Fig. 2(b) shows that the value of λ_N decreases when the number of rewired edges decreases. In fact, this is the reason for disagreement in the results obtained from different order parameters in Fig. 2(a). Throughout

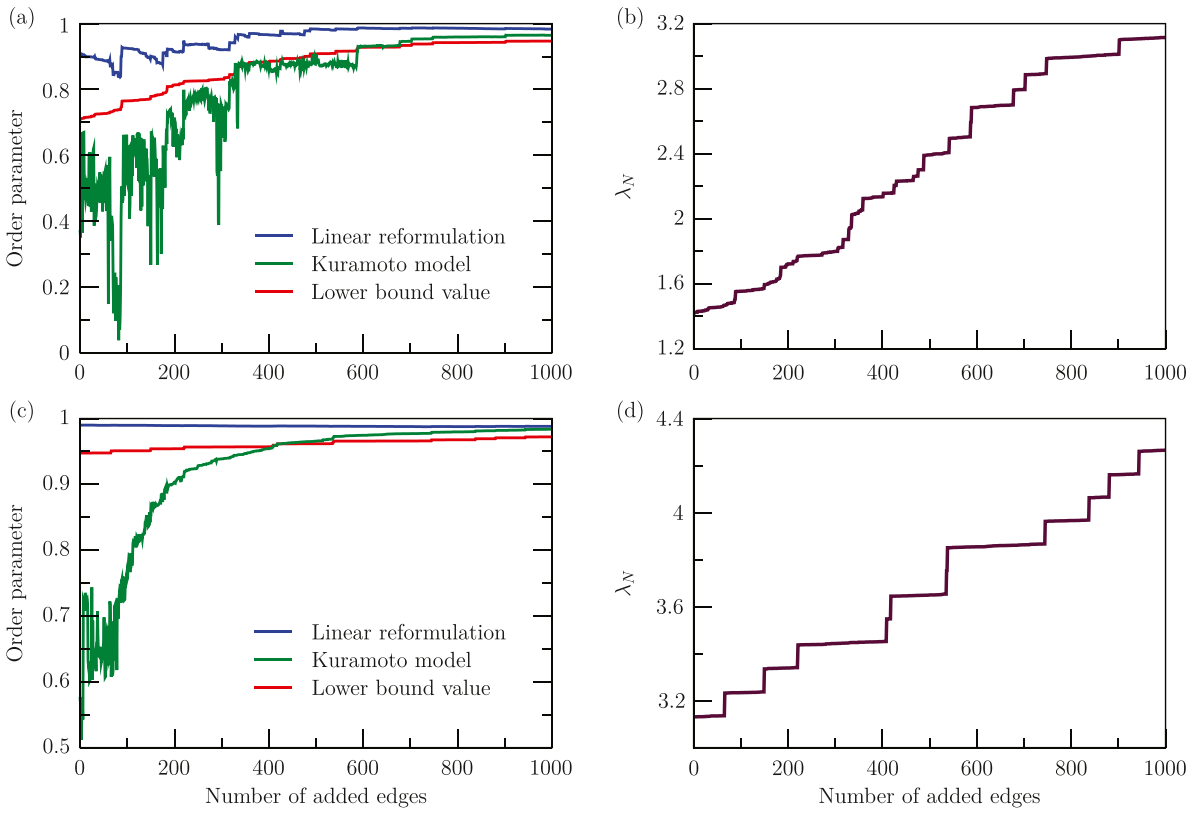

Fig. 3. The linearization error for the Kuramoto model on the networks with two clusters. (a, c) The stationary order parameters for the Kuramoto model and its linear reformulation, and the corresponding lower bound values, versus the number of added edges. (b, d) The largest eigenvalues of the Laplacian matrices versus the number of added edges. The networks have two (Top) Erdős–Rényi or (Bottom) Barabasi–Albert clusters connected by some random edges. Each cluster has $N = 100$ nodes and average degree of 6. The coupling has been considered as $\kappa = 0.1$.

this paper, numerical results for Kuramoto model are obtained by integrating Eq. (3) using Runge–Kutta 4th order method with step size 0.01, and the Kuramoto order parameters have been calculated afterwards. Here, the considered intrinsic frequencies are $\boldsymbol{\omega} = \sqrt{N}\boldsymbol{v}_N$.

A second example might be clustered networks. Fig. 3 shows similar results for networks with two Erdős–Rényi or Barabasi–Albert clusters connected by random edges. We can see that, when the number of links between clusters is low, the corresponding Kuramoto system is not ε-linearizable for small enough ε. Since the largest Laplacian eigenvalue is related to the hub degree, accuracy of approximation is higher for networks with scale-free clusters than networks with Erdős–Rényi clusters.

5. Coupled phase frustrated oscillators

To consider the dynamics of coupled phase oscillators in the presence of phase frustration, suppose that $\boldsymbol{\theta}$ satisfies the system $\langle \boldsymbol{\omega}, \boldsymbol{\theta}_0, K, H \rangle$ where $H_{ij}(\theta) = \sin(\theta - \alpha_{ij})$ for each $1 \le i < j \le N$. The Kuramoto dynamics become

$$\frac{d\theta_i}{dt} = \omega_i + \sum_{j=1}^{N} \kappa_{ij} \sin\big(\theta_j - \theta_i - \alpha_{ij}\big), \tag{16}$$

where $\kappa_{ij} > 0$ is the symmetric coupling strength between oscillators i and j if there is a link between them, and $\kappa_{ij} = 0$ otherwise. Moreover, $\alpha_{ij} = \alpha_{ji}$ for each $1 \le i, j \le N$ and $-\frac{\pi}{2} \le \alpha_{ij} \le \frac{\pi}{2}$ (i.e. $\cos\alpha_{ij} \ge 0$). If all phase differences are small, we can write

$$\sin(\theta_j - \theta_i - \alpha_{ij}) \approx \sin(-\alpha_{ij}) + (\theta_j - \theta_i)\cos\alpha_{ij},$$

and the dynamics become

$$\frac{d\theta_i}{dt} = \omega_i - \sum_{j=1}^{N} \kappa_{ij} \sin\alpha_{ij} + \sum_{j=1}^{N} \kappa_{ij}(\theta_j - \theta_i)\cos\alpha_{ij},$$

for $i = 1, \ldots, N$. Setting

$$\omega_i' = \omega_i - \sum_{j=1}^{N} \kappa_{ij} \sin\alpha_{ij}, \quad 1 \le i \le N$$

and $\kappa_{ij}' = \kappa_{ij}\cos\alpha_{ij}$ for $1 \le i, j \le N$, we obtain

$$\frac{d\theta_i}{dt} = \omega_i' + \sum_{j=1}^{N} \kappa_{ij}'(\theta_j - \theta_i)$$

for $i = 1, \ldots, N$, and so the dynamics become

$$\frac{d\boldsymbol{\theta}}{dt} = \boldsymbol{\omega}' - L'\boldsymbol{\theta} \tag{17}$$

where $\boldsymbol{\omega}' = [\omega_1' \ \ldots \ \omega_N']^T$, and L' is the Laplacian matrix for the weighted adjacency matrix $K' = [\kappa_{ij}']$. Thus, similar to Section 4, if we choose $\boldsymbol{\omega}$ such that $\|L'^{\dagger}(\boldsymbol{\omega} - \sum_{i=1}^{N}\sum_{j=1}^{N}\kappa_{ij}\sin\alpha_{ij}\boldsymbol{e}_i)\| < \varepsilon$ for a small enough $\varepsilon > 0$ where $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_N$ are the vectors of the standard ordered basis for $\mathbb{R}^N$, then by Eq. (12), all phase differences are small, i.e. $\|\boldsymbol{\theta} - \bar{\theta}\mathbf{1}\| < \varepsilon$ and the system $\langle \boldsymbol{\omega}, \boldsymbol{\theta}_0, K, H\rangle$ is ε-linearizable. As the equation is in the form of Eq. (8), its solution is given by

$$\boldsymbol{\theta}(t) = \big(\bar{\omega}'t + \bar{\theta}(0)\big)\mathbf{1} + \sum_{j=2}^{N}\left(\boldsymbol{v}_j\boldsymbol{v}_j^T\left(\frac{-1}{\lambda_j}\boldsymbol{\omega}' + \boldsymbol{\theta}(0)\right)e^{-\lambda_j t} + \frac{1}{\lambda_j}\boldsymbol{v}_j\boldsymbol{v}_j^T\boldsymbol{\omega}'\right),$$

where $\boldsymbol{\omega}' = \boldsymbol{\omega} - \sum_{i=1}^{N}\sum_{j=1}^{N}\kappa_{ij}\sin\alpha_{ij}\boldsymbol{e}_i$, $\bar{\omega}' = \bar{\omega} - \frac{1}{N}\sum_{i,j}\kappa_{ij}\sin\alpha_{ij}$, and $\boldsymbol{v}_1' = \frac{1}{\sqrt{N}}\mathbf{1}, \boldsymbol{v}_2', \ldots, \boldsymbol{v}_N'$ are eigenvectors set of L' corresponding to the eigenvalues $0 = \lambda_1' < \lambda_2' \le \cdots \le \lambda_N'$, respectively. Moreover, Equations 11–(14) still hold with respect to the Laplacian matrix L' and frequency vector $\boldsymbol{\omega}'$.

An interesting special case occurs when all α's are equal and all κ's are also equal, as explored in [38,39]. In this case, we have $L' = (\kappa\cos\alpha)L_u$ and $\boldsymbol{\omega}' = \boldsymbol{\omega} - (\kappa\sin\alpha)\boldsymbol{d}$ where L_u is the (unweighted) Laplacian matrix of the network and $\boldsymbol{d}$ is the vector of degrees for nodes, so Eq. (17) becomes

$$\frac{d\boldsymbol{\theta}}{dt} = \boldsymbol{\omega} - (\kappa\sin\alpha)\boldsymbol{d} - (\kappa\cos\alpha)L_u\boldsymbol{\theta}. \tag{18}$$

Authors in Skardal et al. [38], 39] found that increasing the coupling strength does not lead to perfect synchronization.

We may write

$$\begin{aligned} L'^{\dagger}\boldsymbol{\omega}' &= \frac{1}{\kappa\cos\alpha}L_u^{\dagger}(\boldsymbol{\omega} - (\kappa\sin\alpha)\boldsymbol{d}) \\ &= \frac{1}{\kappa\cos\alpha}L_u^{\dagger}\boldsymbol{\omega} - (\tan\alpha)L_u^{\dagger}\boldsymbol{d}. \end{aligned} \tag{19}$$

When $\kappa \to \infty$, we have

$$L'^{\dagger}\boldsymbol{\omega}' = -(\tan\alpha)L_u^{\dagger}\boldsymbol{d}$$

and hence

$$Lb(\boldsymbol{\omega}') = 1 - \frac{1}{N}\big(\tan^2\alpha\big)\left\|L_u^{\dagger}\boldsymbol{d}\right\|^2,$$

independent from $\boldsymbol{\omega}$. In fact, if $\alpha \neq 0$ and $L_u^{\dagger}\boldsymbol{d} \neq \mathbf{0}$ then we have (see Appendix B)

$$\|L'^{\dagger}\boldsymbol{\omega}'\| \ge \frac{1}{2}|\tan\alpha|\,\|L_u^{\dagger}\boldsymbol{d}\| \quad \text{for} \quad \kappa \ge \frac{2\|L_u^{\dagger}\boldsymbol{\omega}\|}{|\sin\alpha|\,\|L_u^{\dagger}\boldsymbol{d}\|}. \tag{20}$$

This shows that $\|L'^{\dagger}\boldsymbol{\omega}'\|$ may not be less than ε for small-enough values of ε, and therefore using the linearization technique is not well-justified to get a reasonable approximation.

It is stated in [38] that considering the dynamics of Eq. (16) in the strong coupling regime (i.e. $|\theta_j - \theta_i| \ll 1$ for all $1 \le i, j \le N$) and other assumptions that lead to Eq. (18), when $\kappa \to \infty$, perfect synchronization is possible if and only if either $\alpha = 0$ (that may not be the case) or $L_u^{\dagger}\boldsymbol{d} = \mathbf{0}$.

However, the above result is based on assumptions that may not be satisfied. In fact, the error of estimating the original dynamics with the linear form depends on both the approximation error ε in estimation of $\sin\theta$ by θ and the coefficient κ; so, bigger values of κ need smaller values of ε. However, one side of implication still remains true: we show that in the presence of phase frustration, if perfect synchronization is possible, then the network is regular.

Theorem 2. *For a phase frustrated network with constant coupling strength κ and constant phase frustration α, if the phenomenon of perfect phase synchronization for $\kappa \to \infty$ is possible, then the network is regular.*

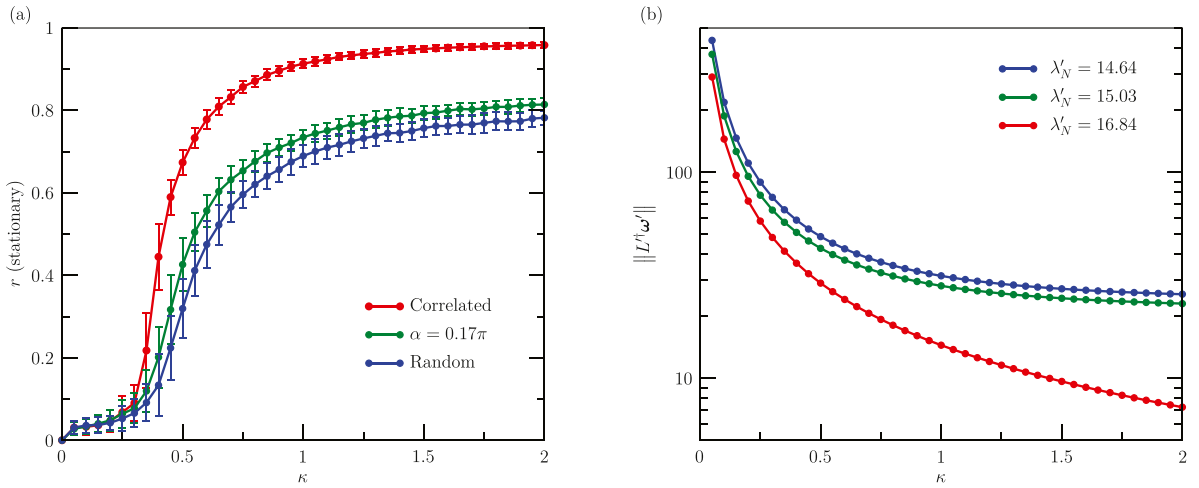

Fig. 4. Synchronization of the Erdős–Rényi networks with different frequency-frustration adjustments. (a) The stationary Kuramoto order parameter and (b) the error of linearization versus couplings for networks with correlated and random frequency-frustration adjustments. Frequencies and phase frustrations are drawn from the power-law distributions such that $g(\omega) \propto \omega^{-2} \in [0, 2\pi]$ and $g'(\alpha) \propto \alpha^{-2} \in [0, \frac{\pi}{2}]$ with mean α equal to 0.17π. The network with fixed phase frustrations equal to the means of the distributions is depicted for comparison. The average value of the largest eigenvalues of the L's for $\kappa = 1$ has been specified for each curve in the plot. Each point is averaged over 50 realizations of networks with $N = 1000$ and average degree 6.

Proof. First assume that when $t \to \infty$, $\theta_i(t) = \theta_j(t)$ for each $1 \le i \le j \le N$. By the assumption of perfect synchronization and Eq. (16), we have

$$\frac{d\theta_i}{dt} = \omega_i - \kappa d_i \sin\alpha.$$

By Eq. (10), we obtain

$$\frac{\omega_i - \omega_j}{\kappa} = (d_i - d_j)\sin\alpha.$$

For any given frequency vector $\boldsymbol{\omega}$ we have

$$\lim_{\kappa\to\infty} \frac{\omega_i - \omega_j}{\kappa} = 0,$$

which shows that $d_i = d_j$ for each $1 \le i < j \le N$ since $\alpha \neq 0$, i.e. the network is regular.■

Theorem 2 shows analytically the connection between structural homogeneity and perfect synchronization in the presence of phase frustration. In case of structural heterogeneity, we show numerically that heterogeneous phase frustration and heterogeneous structure are not correlated.

It has been shown that, in the Kuramoto model, the optimal intrinsic frequency distribution for a network with homogeneous degree distribution, is homogeneous [37]. On the other hand, the distribution of frustrations also affects the synchronizability of the networks. Fig. 4(a) shows that in a network with homogeneous degree distribution and heterogeneous frequency distribution, the correlation between frequencies and frustrations increases the synchronizability of the network. Numerical results exhibit that, the synchronizability of a homogeneous network with adjustments between frequencies and frustrations is even better than the synchronizability of a network with fixed frustration values equal to the mean of the distribution. This result could be further confirmed by calculating the linearization error ($\|L'^{\dagger}\omega'\|$). Indeed, frequency-frustration adjustment leads to the smaller linearization errors (Fig. 4(b)). In different circumstances, for a network with heterogeneous degree distribution and homogeneous frequency distribution, the anti-correlation between degrees and frustration values will increase synchronizabilty (see Fig. 5). One of the reasons for the higher synchronizability of the adjusted networks with heterogeneous phase frustrations is the rise of λ'_N, which flows from the heterogeneity that phase frustration introduces to the systems.

6. Conclusion and future work

Despite the notion of the "strong coupling regime" that is very frequent and assumed in the literature, we have considered the notion of ε-linearizability for Kuramoto models and determine a necessary condition under which a Kuramoto system $\langle \boldsymbol{\omega}, \boldsymbol{\theta}_0, K, H\rangle$ is ε-linearizable for a sufficiently small ε. We studied lower bounds for order parameter that measures phase-lock synchronization in complex oscillator networks. One approach to deal with a system of ordinary differential equations of Kuramoto model is to approximate it by its linear form. The importance of linearization error was shown, and

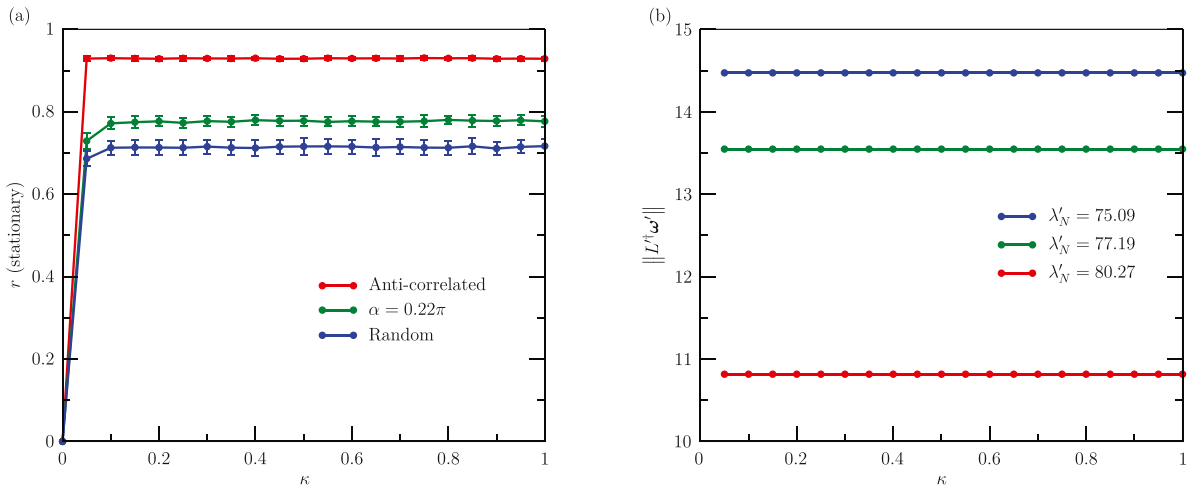

Fig. 5. Synchronization of the Barabasi–Albert networks with different degree-frequency adjustments. (a) The stationary order parameter and (b) the error of linearization versus couplings for networks with anti-correlated and random degree-frustration adjustments. All natural frequencies are equal to one and phase frustrations are drawn from power-law distributions such that $g'(\alpha) \propto \alpha^{-3} \in [\frac{\pi}{8}, \frac{\pi}{2}]$ with mean α equal to 0.22π. The network with fixed phase frustrations equal to the means of the distributions is depicted for comparison. The average value of the largest eigenvalues of the L's for $\kappa = 1$ has been specified for each curve in the plot. Each point is averaged over 50 realizations of the networks with $N = 1000$, and average degree 6.

we proved that if linear approximation to the Kuramoto dynamics is reasonable, then an appropriate multiple of the eigenvector corresponding to the maximum eigenvalue of Laplacian matrix of its network is an optimum frequency vector over all vectors with standard deviation 1. Moreover, we proved that perfect phase synchronization for $\kappa \to \infty$ for a globally coupled network with frustration implies the regularity of the network. Moreover, for a locally coupled network with frustration, we examined the dependence between frustration values and degrees or frequencies on the synchronization optimization.

The main results reported in this paper can be summarized as follows:

- A necessary condition for ε-linearizability for globally or locally coupled networks is given.
- it is proved mathematically that an appropriate coefficient of the eigenvector corresponding to the largest eigenvalue of the Laplacian matrix optimizes the lower bound for order parameter among frequency vectors with average zero and standard deviation 1 under the assumption of ε-linearizability.
- It was shown through numerical simulations that it is not possible to ε-linearize the Kuramoto system with a sufficiently small ε for a family of networks on the space of frequency vectors with average zero and standard deviation 1.
- We proved that under the assumption of perfect phase synchronization for $\kappa \to \infty$ in the Kuramoto system with phase frustration, the network must be regular.
- The effect of correlation between phase frustrations and degrees (or frequencies) on the order parameter of systems is shown.

Further work can consider providing more general techniques to find exact and approximate solutions for the Kuramoto system and extending the results in this paper to wider classes of 2π-periodic coupling functions on directed or signed networks.

Higher order Kuramoto models are also important. For example, the second-order Kuramoto model has been used to investigate power grids. Developing a new framework to study these systems, together with a new measure to quantify the synchronizability, is another line of research for future research.

Appendix A. Weighted adjacency matrices

Suppose that $G = \langle V, E \rangle$ is a weighted graph whose weighted adjacency matrix is $K = [\kappa_{ij}]$. We adopt Propositions 4.3, 4.8, and 4e from [49] to prove some facts about the rank and eigenvalues of the (weighted) Laplacian matrix of G.

Put some arbitrary direction on the edges of G, and let C be the matrix whose elements are defined by

$$c_{ij} = \begin{cases} \sqrt{\kappa_{il}} & \text{if } v_i \text{ is the initial vertex of the edge } e_j = \{v_i, v_l\} \\ -\sqrt{\kappa_{il}} & \text{if } v_i \text{ is the final vertex of the edge } e_j = \{v_l, v_i\} \\ 0 & \text{if } v_i \notin e_j. \end{cases}$$

Lemma 2. *If G is connected, then* $\operatorname{rank}(C) = N - 1$.

Proof. As $C^T\mathbf{1} = \mathbf{0}$, we have $\text{rank}(C) \leq N-1$. Let C_i be the i-th row of C for $i = 1, \ldots, N$, and suppose that some linear combination of C_i's is zero; i.e.

$$\sum_{i=1}^{N} \alpha_i C_i = 0.$$

Since each column of C has exactly two nonzero elements and these elements have the same value with different signs, if a coefficient α_k is nonzero, then there should be another index l such that α_l is also nonzero and equal to α_k. Since G is connected, it follows that all C_i's have the same nonzero coefficient and hence the only nontrivial linear combination that is zero is $C_1 + \cdots + C_N = 0$, which shows that $\text{rank}(C) = N-1$.■

Let $d_i = \sum_{j=1}^{N} \kappa_{ij}$ be the sum of weights of edges that are connected to vertex v_i and define Δ to be the diagonal matrix whose diagonal elements are $d_1, \ldots, d_N$. With some simple calculations, we see that

$$C_i \cdot C_i^T = \sum_{k=1}^{N} \kappa_{ik} = d_i,$$

and if $i \neq j$, the only column which both C_i and C_j can have nonzero elements in it is the column that corresponds to the edge $\{v_i, v_j\}$, and so we have

$$C_i \cdot C_j^T = -\kappa_{ij}.$$

Putting these together, we have proved the following lemma.

Lemma 3. $CC^T = \Delta - K$.

Note that CC^T does not depend to the direction assigned to the edges of G.
Since CC^T is a symmetric matrix, it has N real eigenvalues $\lambda_1 \leq \cdots \leq \lambda_N$ with orthonormal eigenvectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_N$.

Lemma 4. *For the N eigenvalues, we have* $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N$. *Moreover, if G is connected then* $\lambda_2 > 0$.

Proof. As $CC^T\boldsymbol{v}_i = \lambda_i \boldsymbol{v}_i$, we have

$$\begin{aligned} \|C^T\boldsymbol{v}_i\|^2 &= \boldsymbol{v}_i^T CC^T \boldsymbol{v}_i \\ &= \lambda_i \boldsymbol{v}_i^T \boldsymbol{v}_i \\ &= \lambda_i \|\boldsymbol{v}_i\|^2, \end{aligned}$$

which shows that $\lambda_i \geq 0$.
If G is connected, then $\text{rank}(L) = N-1$ and hence there is exactly one zero eigenvalue.■

Now, we can summarize all of the above results in the following theorem.

Theorem 3. *Suppose that* $G = \langle V, E \rangle$ *is a weighted graph whose weighted adjacency matrix is* $K = [\kappa_{ij}]$. *Let* $d_i = \sum_{j=1}^{N} \kappa_{ij}$ *be the sum of weights of edges that are connected to vertex* v_i *and define* Δ *to be the diagonal matrix whose diagonal elements are* $d_1, \ldots, d_N$, *and let L be the Laplacian matrix defined by* $L = \Delta - K$. *Then*

1. $\text{rank}(L) = N - c$, *where c is the number of connected components of G.*
2. *L is symmetric with N real eigenvalues* $0 = \lambda_1 = \cdots = \lambda_c < \lambda_{c+1} \leq \cdots \leq \lambda_N$.

Theorem 4 (Adopted from [46, p.194])**.** *All eigenvalues of the Laplacian matrix of a weighted network* $G = \langle V, E \rangle$ *are less than or equal to*

$$\sqrt{\frac{\kappa_{\max}}{\kappa_{\min}}} \times \max\{d_i + d_j : v_i v_j \in E\},$$

where

$$\begin{aligned} \kappa_{\max} &= \max\left\{\kappa_{ij} : 1 \leq i, j \leq N\right\}, \\ \kappa_{\min} &= \min\left\{\kappa_{ij} : 1 \leq i, j \leq N,\ \kappa_{ij} \neq 0\right\}. \end{aligned}$$

Proof. By Lemma 3, $L = CC^T$. Let $\boldsymbol{y} = C^T\boldsymbol{v}_N$, thus $C^TC\boldsymbol{y} = \lambda_N \boldsymbol{y}$. Choose an edge $e = \{v_h, v_k\}$ such that

$$|y_e| = \max\{|y_f| : f \in E\}.$$

Hence,

$$\lambda_N y_e = \sum_{f \in E} \Big(\sum_{i \in V} c_{ie} c_{if} \Big) y_f,$$

where $c_{ie}c_{if} \neq 0$ if and only if the vertex i is common to the edges e and f. Thus

$$\lambda_N y_e = \sum_{f\in E} c_{he}c_{hf}y_f + \sum_{f\in E} c_{ke}c_{kf}y_f.$$

As a result,

$$\lambda_N|y_e| \leq |y_e|\sum_{j=1}^{N}\sqrt{\kappa_{hk}}\sqrt{\kappa_{hj}} + |y_e|\sum_{j=1}^{N}\sqrt{\kappa_{hk}}\sqrt{\kappa_{kj}}.$$

Now, we may write

$$\begin{aligned}
\lambda_N &\leq \textstyle\sum_{j=1}^{N}\sqrt{\kappa_{hk}}\sqrt{\kappa_{hj}} + \sum_{j=1}^{N}\sqrt{\kappa_{hk}}\sqrt{\kappa_{kj}} \\
&= \kappa_{\min}\times\sqrt{\tfrac{\kappa_{hk}}{\kappa_{\min}}}\times\left(\textstyle\sum_{j=1}^{N}\sqrt{\tfrac{\kappa_{hj}}{\kappa_{\min}}} + \sum_{j=1}^{N}\sqrt{\tfrac{\kappa_{kj}}{\kappa_{\min}}}\right) \\
&\leq \kappa_{\min}\times\sqrt{\tfrac{\kappa_{hk}}{\kappa_{\min}}}\times\left(\textstyle\sum_{j=1}^{N}\tfrac{\kappa_{hj}}{\kappa_{\min}} + \sum_{j=1}^{N}\tfrac{\kappa_{kj}}{\kappa_{\min}}\right) \\
&= \sqrt{\tfrac{\kappa_{hk}}{\kappa_{\min}}}\times\left(\textstyle\sum_{j=1}^{N}\kappa_{hj} + \sum_{j=1}^{N}\kappa_{kj}\right) \\
&= \sqrt{\tfrac{\kappa_{hk}}{\kappa_{\min}}}\times(d_h + d_k).
\end{aligned}$$

■

Appendix B. Supplementary and missing proofs

Proof of Equation (2).. First, we re-write Eq. (1) by Laplacian Matrix:

$$\begin{aligned}
\tfrac{d\theta_i}{dt} &= \omega_i + \kappa\textstyle\sum_{j=1}^{N}A_{ij}H\left(\theta_j-\theta_i\right) \\
&= \omega_i + \kappa\textstyle\sum_{j=1}^{N}\delta_{ij}d_iH\left(\theta_j-\theta_i\right) - \kappa\sum_{j=1}^{N}L_{ij}H\left(\theta_j-\theta_i\right) \\
&= \omega_i + \kappa d_iH(0) - \kappa\textstyle\sum_{j=1}^{N}L_{ij}H\left(\theta_j-\theta_i\right).
\end{aligned}$$

By Taylor's theorem, we have $H(\theta_j-\theta_i)\approx H(0)+H'(0)(\theta_j-\theta_i)$. Since $\sum_{j=1}^{N}L_{ij}=0$ for all $1\leq i\leq N$, we obtain:

$$\begin{aligned}
\tfrac{d\theta_i}{dt} &= \omega_i + \kappa d_iH(0) - \kappa\textstyle\sum_{j=1}^{N}L_{ij}H\left(\theta_j-\theta_i\right) \\
&\approx \omega_i + \kappa d_iH(0) - \kappa\textstyle\sum_{j=1}^{N}L_{ij}\left(H(0)+H'(0)\left(\theta_j-\theta_i\right)\right) \\
&= \omega_i + \kappa d_iH(0) - \kappa\textstyle\sum_{j=1}^{N}L_{ij}\left(H(0)-H'(0)\theta_i\right) - \kappa\sum_{j=1}^{N}L_{ij}\left(H'(0)\theta_j\right) \\
&= \omega_i + \kappa d_iH(0) - \kappa\left(H(0)-H'(0)\theta_i\right)\textstyle\sum_{j=1}^{N}L_{ij} - \kappa\sum_{j=1}^{N}L_{ij}\left(H'(0)\theta_j\right) \\
&= \omega_i + \kappa d_iH(0) - \kappa\textstyle\sum_{j=1}^{N}L_{ij}\left(H'(0)\theta_j\right).
\end{aligned}$$

■

Proof of Equation (4)..

$$\begin{aligned}
R(t) &= \frac{1}{N^2}\left|\sum_{j=1}^{N}e^{i\theta_j(t)}\right|^2 \\
&= \frac{1}{N^2}\left|\sum_{j=1}^{N}\left(\cos\left(\theta_j(t)\right)+i\sin\left(\theta_j(t)\right)\right)\right|^2 \\
&= \frac{1}{N^2}\left(\left(\sum\nolimits_j\cos\theta_j\right)^2+\left(\sum\nolimits_j\sin\theta_j\right)^2\right) \\
&= \frac{1}{N^2}\left(\sum\nolimits_{i,j}\cos\theta_i\cos\theta_j+\sum\nolimits_{i,j}\sin\theta_i\sin\theta_j\right) \\
&= \frac{1}{N^2}\sum\nolimits_{i,j}\cos(\theta_i-\theta_j).
\end{aligned}$$

■

Proof of Proposition 1. Since $|\sin(\theta_i-\theta_j)|<|\theta_i(t)-\theta_j(t)|<\varepsilon\leq 1$ for all $t>t_s$, by Eq. (4), we have

$$\begin{aligned}
R(t) &= \tfrac{1}{N^2}\textstyle\sum_{i,j}\cos\left(\theta_i-\theta_j\right) \\
&\geq \tfrac{1}{N^2}\textstyle\sum_{i,j}\sqrt{1-\varepsilon^2} \\
&= \sqrt{1-\varepsilon^2}.
\end{aligned}$$

■

Proof of Proposition 2. Since $\cos x \geq 1 - \frac{1}{2}x^2$ for all $x \in \mathbb{R}$, we obtain

$$\begin{aligned} R(t) &= \tfrac{1}{N^2}\sum_{i,j}\cos\left(\theta_i(t)-\theta_j(t)\right) \\ &\geq \tfrac{2N^2-\sum_{i,j}\left(\theta_i(t)-\theta_j(t)\right)^2}{2N^2} \\ &= \tfrac{2N^2-\sum_{i,j}\left(\theta_i(t)-\overline{\theta}(t)+\overline{\theta}(t)-\theta_j(t)\right)^2}{2N^2} \\ &\geq \tfrac{2N^2-\sum_{i,j}\left(\theta_i(t)-\overline{\theta}(t)\right)^2-\sum_{i,j}\left(\overline{\theta}(t)-\theta_j(t)\right)^2}{2N^2} \\ &= \tfrac{2N^2-2N\sum_{i}\left(\theta_i(t)-\overline{\theta}(t)\right)^2}{2N^2} \\ &= 1-\tfrac{1}{N}\|\boldsymbol{\theta}(t)-\overline{\theta}(t)\mathbf{1}\|^2. \end{aligned}$$

■

Proof of Proposition 3.. It is easy to check that $\boldsymbol{\varphi}(t)$ is a solution for the system

$$\frac{d\varphi_i}{dt} = \omega_i + c + \sum_{j=1}^{N}\kappa_{ij}H_{ij}(\varphi_j-\varphi_i),$$

and $\boldsymbol{\varphi}(0) = \boldsymbol{\theta}(0)$. Moreover, if we set $\boldsymbol{\psi}(t) = \boldsymbol{\theta}(\frac{1}{c}t)$ and $\kappa'_{ij} = \kappa_{ij}/c$ for $1 \leq i, j \leq N$ then we have

$$c\frac{d\psi_i}{dt} = \omega_i + \sum_{j=1}^{N}\kappa_{ij}H_{ij}(\psi_j-\psi_i)$$

or

$$\frac{d\psi_i}{dt} = \omega_i/c + \sum_{j=1}^{N}\kappa'_{ij}H_{ij}(\psi_j-\psi_i),$$

which shows that $\boldsymbol{\psi}(t)$ satisfies system $\langle \frac{1}{c}\boldsymbol{\omega}, \boldsymbol{\theta}_0, \frac{1}{c}K, H\rangle$.■

Proof of Equation (9).. In Eq. (8), by using the standard Laplace transformation operator $\mathcal{L}\{\}$, we have

$$\begin{bmatrix}\mathcal{L}\{\frac{d\theta_1}{dt}\}\\ \vdots \\ \mathcal{L}\{\frac{d\theta_N}{dt}\}\end{bmatrix} = \begin{bmatrix}\mathcal{L}\{\omega_1\}\\ \vdots \\ \mathcal{L}\{\omega_N\}\end{bmatrix} - L\begin{bmatrix}\mathcal{L}\{\theta_1(t)\}\\ \vdots \\ \mathcal{L}\{\theta_N(t)\}\end{bmatrix}$$

or,

$$s\begin{bmatrix}\mathcal{L}\{\theta_1(t)\}\\ \vdots \\ \mathcal{L}\{\theta_N(t)\}\end{bmatrix} - \begin{bmatrix}\theta_1(0)\\ \vdots \\ \theta_N(0).\end{bmatrix} = \frac{1}{s}\begin{bmatrix}\omega_1\\ \vdots \\ \omega_N\end{bmatrix} - L\begin{bmatrix}\mathcal{L}\{\theta_1(t)\}\\ \vdots \\ \mathcal{L}\{\theta_N(t)\}\end{bmatrix}$$

which can be written in a more compact form as

$$(sI+L)\mathcal{L}\{\boldsymbol{\theta}\} = \frac{1}{s}\boldsymbol{\omega} + \boldsymbol{\theta}(0)$$

for $s > 0$, where $\mathcal{L}\{\boldsymbol{\theta}\} = [\mathcal{L}\{\theta_1(t)\} \ \ldots \ \mathcal{L}\{\theta_N(t)\}]^T$. We see that the eigenvalues of $sI+L$ are $s+\lambda_1, \ldots, s+\lambda_N$ which are all positive since $s > 0$. It follows that $sI+L$ is invertibe and

$$\begin{aligned}(sI+L)^{-1} &= P\begin{bmatrix}\frac{1}{s} & & & 0\\ & \frac{1}{s+\lambda_2} & & \\ & & \ddots & \\ 0 & & & \frac{1}{s+\lambda_N}\end{bmatrix}P^T \\ &= \frac{1}{s}\boldsymbol{v}_1\boldsymbol{v}_1^T + \frac{1}{s+\lambda_2}\boldsymbol{v}_2\boldsymbol{v}_2^T + \ldots + \frac{1}{s+\lambda_N}\boldsymbol{v}_N\boldsymbol{v}_N^T\end{aligned}$$

that gives

$$\begin{aligned}\mathcal{L}\{\boldsymbol{\theta}\} &= \tfrac{1}{s^2}\boldsymbol{v}_1\boldsymbol{v}_1^T\boldsymbol{\omega} + \tfrac{1}{s}\boldsymbol{v}_1\boldsymbol{v}_1^T\boldsymbol{\theta}(0) + \sum_{j=2}^{N}\left(\tfrac{1}{s(s+\lambda_j)}\boldsymbol{v}_j\boldsymbol{v}_j^T\boldsymbol{\omega} + \tfrac{1}{s+\lambda_i}\boldsymbol{v}_j\boldsymbol{v}_j^T\boldsymbol{\theta}(0)\right)\\ &= \tfrac{1}{s^2}\boldsymbol{v}_1\boldsymbol{v}_1^T\boldsymbol{\omega} + \tfrac{1}{s}\boldsymbol{v}_1\boldsymbol{v}_1^T\boldsymbol{\theta}(0) + \sum_{j=2}^{N}\left(\tfrac{1}{\lambda_j}\left(\tfrac{1}{s} - \tfrac{1}{s+\lambda_j}\right)\boldsymbol{v}_j\boldsymbol{v}_j^T\boldsymbol{\omega} + \tfrac{1}{s+\lambda_j}\boldsymbol{v}_j\boldsymbol{v}_j^T\boldsymbol{\theta}(0)\right)\\ &= \tfrac{1}{s^2}\boldsymbol{v}_1\boldsymbol{v}_1^T\boldsymbol{\omega} + \tfrac{1}{s}\boldsymbol{v}_1\boldsymbol{v}_1^T\boldsymbol{\theta}(0) + \sum_{j=2}^{N}\left(\tfrac{1}{s+\lambda_j}\boldsymbol{v}_j\boldsymbol{v}_j^T\left(\tfrac{-1}{\lambda_j}\boldsymbol{\omega} + \boldsymbol{\theta}(0)\right) + \tfrac{1}{\lambda_j}\times\tfrac{1}{s}\boldsymbol{v}_j\boldsymbol{v}_j^T\boldsymbol{\omega}\right).\end{aligned}$$

Using the inverse Laplace transform, we obtain

$$\begin{aligned}\boldsymbol{\theta}(t) &= t\boldsymbol{v}_1\boldsymbol{v}_1^T\boldsymbol{\omega} + \boldsymbol{v}_1\boldsymbol{v}_1^T\boldsymbol{\theta}(0) + \sum_{j=2}^{N}\left(\boldsymbol{v}_j\boldsymbol{v}_j^T\left(\tfrac{-1}{\lambda_j}\boldsymbol{\omega} + \boldsymbol{\theta}(0)\right)e^{-\lambda_j t} + \tfrac{1}{\lambda_j}\boldsymbol{v}_j\boldsymbol{v}_j^T\boldsymbol{\omega}\right)\\ &= \left(\overline{\omega}t + \overline{\theta}(0)\right)\mathbf{1} + \sum_{j=2}^{N}\left(\boldsymbol{v}_j\boldsymbol{v}_j^T\left(\tfrac{-1}{\lambda_j}\boldsymbol{\omega} + \boldsymbol{\theta}(0)\right)e^{-\lambda_j t} + \tfrac{1}{\lambda_j}\boldsymbol{v}_j\boldsymbol{v}_j^T\boldsymbol{\omega}\right).\end{aligned}$$

■

Proof of Equation (20).. By Eq. (19) and the triangle inequality,

$$\|L'^{\dagger}\boldsymbol{\omega}'\| \geq |\tan\alpha|\,\|L_u^{\dagger}\boldsymbol{d}\| - \frac{1}{\kappa|\cos\alpha|}\|L_u^{\dagger}\boldsymbol{\omega}\|.$$

Thus, for $\alpha \neq 0$ and $\|L_u^{\dagger}\boldsymbol{d}\| \neq 0$, we have

$$|\tan\alpha|\,\|L_u^{\dagger}\boldsymbol{d}\| - \frac{1}{\kappa|\cos\alpha|}\|L_u^{\dagger}\boldsymbol{\omega}\| \geq \frac{1}{2}|\tan\alpha|\,\|L_u^{\dagger}\boldsymbol{d}\|$$

if and only if

$$\kappa \geq \frac{2\|L_u^{\dagger}\boldsymbol{\omega}\|}{|\sin\alpha|\,\|L_u^{\dagger}\boldsymbol{d}\|}.$$

■

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