Fault-Tolerant Spanners in Networks with Symmetric Directional Antennas

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Abstract. Let P be a set of points in the plane, each equipped with a directional antenna that can cover a sector of angle α and range r. In the symmetric model of communication, two antennas u and v can communicate to each other, if and only if v lies in u's coverage area and vice versa. In this paper, we introduce the concept of *fault-tolerant spanners* for directional antennas, which enables us to construct communication networks that retain their connectivity and spanning ratio even if a subset of antennas are removed from the network. We show how to orient the antennas with angle α and range r to obtain a k-fault-tolerant spanner for any positive integer k. For $\alpha \geq \pi$, we show that the range 13 for the antennas is sufficient to obtain a k-fault-tolerant 3-spanner. For $\pi/2 < \alpha < \pi$, we show that using range $6\delta + 19$ for $\delta = \lceil 4/|\cos\alpha| \rceil$, one can direct antennas so that the induced communication graph is a k-fault-tolerant 7-spanner.

1 Introduction

Omni-directional antennas, whose coverage area are often modelled by a disk, have been traditionally employed in wireless networks. However, in many recent applications, omni-directional antennas have been replaced by directional antennas, whose coverage region can be modelled as a sector with an angle α and a radius r (also called transmission range), where the orientation of antennas can vary among the nodes of the network. The point is that by a proper orientation of directional antennas, one can generate a network with lower radio wave overlapping and higher security than the traditional networks with omni-directional antennas [4].

There are two main models of communication in networks with directional antennas. In the *asymmetric* model, each antenna has a directed link to any node that lies in its coverage area. In the *symmetric* model, there exists a link between two antennas u and v, if and only if u lies in the coverage area of v, and v lies in the coverage area of u. The symmetric model of communication is more practical, especially in networks where two nodes must handshake to each other before transmitting data [6].

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In this paper, we consider the symmetric model for communication in directional antennas, and study two properties of the communication graphs: *k-connectivity* and *spanning ratio*. A network is *k-connected* if it remains connected after removing or destroying any $k - 1$ of its nodes. Furthermore, if after some failure of nodes, it still has some desirable properties, we say that the network is *fault-tolerant*. Therefore, the fault-tolerance property is more general than the connectivity. A network is called a *spanner*, if there is a short path between any pairs of nodes, within a guaranteed ratio to the shortest paths between those nodes in an underlying base graph. This ratio is called the *stretch factor*. A *fault-tolerant spanner* has the property that when a small number of nodes fail, the remaining network still contains short paths between any pair of nodes. (See [14] for an overview of the properties of geometric spanner networks.)

Related Work. The problem of orienting directional antennas to obtain a strongly connected network was first studied by Caragiannis *et al.* [5] in the asymmetric model. They showed that the problem is NP-hard for $\alpha < 2\pi/3$, and presented a polynomial time algorithm for $\alpha \geq 8\pi/5$ with optimal radius. The problem was later studied for other values of α, and approximation algorithms were provided to minimize the transmission range of connected networks [1,7]. However, the communication graphs obtained from these algorithm could have a very large stretch factor, such as $O(n)$, compared to the original unit disk graph (i.e., the omni-directional graph of radius 1). Therefore, subsequent research was shifted towards finding a proper orientation such that the resulting graph becomes a t-hop spanner [4,11]. In a *t-hop spanner*, the number of hops (i.e., links) in a shortest link path between any pair of nodes is at most t times the number of hops in the shortest link path between those two nodes in the base graph, which happens to be a unit disk graph in this case.

The connectivity of communication graphs in the symmetric model was first studied by Ben-Moshe *et al.* [3] in a limited setting where the orientation of antennas were chosen from a fixed set of directions. Carmi *et al.* [6] later considered the general case, and proved that for $\alpha \geq \pi/3$, it is always possible to orient antennas so that the induced graph is connected. In their presented algorithm, the radius of the antennas were related to the diameter of the nodes. Subsequent work considered the stretch factor of the communication graph. Aschner *et al.* [2] studied the problem for $\alpha = \pi/2$ and obtained a symmetric connected network with radius $14\sqrt{2}$ and a stretch factor of 8, assuming that the unit disk graph of the nodes is connected. Recently, Dobrev *et al.* [8] proved that for $\alpha < \pi/3$ and radius one, the problem of connectivity in the symmetric model is also NP-hard. They also showed how to construct spanners for various values of $\alpha \geq \pi/2$. A summary of the current records for the radius and the stretch factor of the communication graphs in the symmetric model is presented in Table 1.

The problem of k-connectivity in wireless networks has been also studied in the literature, mostly for omni-directional networks [12,13], where the objective is to assign transmission range such that the network can sustain fault nodes and remain connected. The stretch factor of the constructed network is also

Table 1. Summary of the previous results for netwroks with symmetric directional antennas. In all these results, the unit disk graph of the nodes (antennas) is assumed to be connected. Here, $\delta = \sqrt{3 - 2\cos\alpha(1 + 2\sin\frac{\alpha}{2})}$.

Angle of antenna	Stretch factor	Radius	Ref.
$\pi/2$	8	$14\sqrt{2}$	[2]
$\pi/2$	7	33	[9]
	5	718	
$\pi/2 \leq \alpha < 2\pi/3$	9	10	[8]
$2\pi/3 \leq \alpha < \pi$	–	5	
$2\pi/3 \leq \alpha < \pi$	6	6	
$\alpha \geq \pi$	–	$\max(2, 2\sin\frac{\alpha}{2} + 1)$	
$\alpha \geq \pi$	3	$\max(2, 2\sin\frac{\alpha}{2} + \delta)$	

studied in some limited settings. In [10], a setting is studied where antennas are on a unit segment or a unit square, and a sufficient condition is obtained on the angle of directional antennas so that the energy consumption of the k-connected networks is lower when using directed rather than omni-directed antennas. In [15], a tree structure is built on directed antennas, and a fault-tolerance property is maintained by adding additional links to tolerate failure in limited cases, namely, when only a node or a pair of adjacent nodes fail.

Our Results. In this paper, we study the problem of finding fault-tolerant spanners in networks with symmetric directional antennas. The problem is formally defined as follows. Given a set P of n points in the plane, place antennas with angle α and radius r on P, so that the resulting communication graph is a k-fault-tolerant t-spanner. A graph G on the vertex set P is a *k-fault-tolerant t-spanner*, if after removing any subset $S \subseteq P$ of nodes with $|S| < k$, the resulting graph $G \setminus S$ is a t-spanner of the unit disk graph of P. In the rest of the paper, we assume that the unit distance is sufficiently large to ensure that the unit disk graph of P is k-connected. To the best of our knowledge, this is the first time that fault-tolerance is studied in networks with symmetric directional antennas.

We show that for any $\alpha \geq \pi$, we can place antennas with angle α and radius 9, such that the resulting communication graph is k-connected. Moreover, we show that by increasing the radius to 13, we can guarantee that the resulting graph is a k-fault-tolerant 3-spanner. When $\pi/2 < \alpha < \pi$, we consider two cases depending on whether the distribution of antennas is sparse or dense. We prove that for sparse distribution, we can place antennas with angle α and radius $6\delta + 19$, where $\delta = \lceil 4/|\cos\alpha| \rceil$, such that the resulting communication graph is a k-fault-tolerant 7-spanner. Moreover, for dense distribution, we prove that our algorithm yields a k-fault-tolerant 4-spanner using radius δ. Our results are summarized in Table 2.

We recall that the k-connectivity of the unit disk graph is assumed in the rest of the paper. In other words, we compared the radius and stretch factor of our k-connected directional network to those of a k-connected omni-directional network. While this assumption is reasonable, it is possible to relax it, and only assume the connectivity of the unit disk graph, which is the minimum requirement assumed in the related (non-fault-tolerant) work. If we replace the k-connectivity assumption with 1-connectivity, the radius and stretch factor of our constructed network is increased by a factor of k, as explained in Sect. 5.

Table 2. Summary of our results for netwroks with symmetric directional antennas. In these results, the unit disk graph of the nodes is assumed to be k-connected. Here, $\delta = \lceil 4/|\cos\alpha|\rceil$.

Angle of antenna	Stretch factor	Radius	Ref.
$\alpha \geq \pi$	–	9	Theorem 1
$\alpha \geq \pi$	3	13	Theorem 2
$\pi/2 < \alpha < \pi$ (sparse)	7	$6\delta + 19$	Theorem 3
$\pi/2 < \alpha < \pi$ (dense)	4	δ	Theorem 3

2 Preliminaries

Let P be a set of points in the plane, and G be a graph on the vertex set P. For two points $p, q \in P$, we denote by $\delta_G(p, q)$ the shortest hop (link) distance between p and q in G. If the graph G is clear from the context, we simply write $\delta(p, q)$ instead of $\delta_G(p, q)$. Throughout this paper, the *length* of a path in a graph refers to the number of edges on that path. For two points p and q in the plane, the Euclidean distance between p and q is denoted by $\|pq\|$.

Let $\mathcal{B}(c, r)$ denote a (closed) disk of radius r centered at c. We define $\mathcal{A}(c, r) \equiv \mathcal{B}(c, r) - \mathcal{B}(c, r-1)$ to be an *annulus* of width 1 enclosed by two concentric circles of radii $r - 1$ and r, centered at c. Note that by our definition, $\mathcal{A}(c, r)$ is open from its inner circle, and is closed from the outer circle.

A graph G is k-connected, if removing any set of at most $k-1$ vertices leaves G connected. Given a point set P, we denote by UDG(P) the unit disk graph defined by the set of disks $\mathcal{B}(p, 1)$ for all $p \in P$. We say that P is *k-connected*, if UDG(P) is k-connected. Let $G = $ UDG(P). A graph H on the vertex set P is a *t-spanner* of G, if for any two vertices u and v in G, we have $\delta_H(u, v) \leq t \cdot \delta_G(u, v)$. We say that the subgraph $H \subseteq G$ is a *k-fault-tolerant t-spanner* of G, if for all sets $S \subseteq P$ with $|S| < k$, the graph $H \setminus S$ is a t-spanner of $G \setminus S$.

Fact 1. *Let G and H be two k-connected graphs, and E be a set of edges between the vertices of G and H. If E contains a matching of size k, then the graph $G \cup H \cup E$ is k-connected.*

Fact 2. *Let G be a k-connected graph, and v be a new vertex adjacent to at least k vertices of G. Then $G + v$ is k-connected.*

Lemma 1. *Let P be a k-connected point set, and r be a positive integer. If $|P| \geq rk$, then for any point $p \in P$, $\mathcal{B}(p, r)$ contains at least rk points of P.*

Proof. Fix a point p, and let q be the furthest point from p in P. If $\|pq\| \leq r$, then the disk $P \subseteq \mathcal{B}(p, r)$, and we are done. Otherwise, consider the annuli $A_i = \mathcal{A}(p, i)$ for $1 \leq i \leq r + 1$, and let $A_0 = \{p\}$. Each A_i must be non-empty, because otherwise, p is disconnected from q in UDG(P). Now, we claim that each A_i, for $1 \leq i \leq r$, contains at least k points. Otherwise, if $|A_i| < k$ for some $1 \leq i \leq r$, then removing the points of A_i disconnects A_{i-1} from A_{i+1}, contradicting the fact that P is k-connected. □

3 Antennas with $\alpha \geq \pi$

In this section, we present our algorithm for orienting antennas with angle at least π. The main ingredient of our method is a partitioning algorithm which we describe below.

Partitioning Algorithm. The following algorithm builds a graph H on the input point set P. The graph will induce a partitioning on the input set, as described in Lemma 2. In the following algorithm, p is an arbitrary point of P, and r is a positive integer.

Algorithm 1. PARTITION(P, p, r)

```
1: add vertex p to graph H
2: P = P \ B(p, 2r)
3: while ∃q ∈ P ∩ B(p, 2r + 1) do
4:     PARTITION(P, q, r)
5:     add edge (p, q) to graph H
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Lemma 2. *Let P be a k-connected point set, p be an arbitrary point in P, and $|P| \geq kr$ for a positive integer r. Let $H = (V, E)$ be the graph obtained from* PARTITION(P, p, r). *For each $v \in V$, we define $Q_v = P \cap \mathcal{B}(v, r)$. Moreover, we define F_v to be the set of all points in $P \setminus \cup_{u \in V} Q_u$ closer to v than any other point in V (ties broken arbitrarily). Then the followings hold:*

(a) H is connected, and for each edge $(u, v) \in E$, $2r < \|uv\| \leq 2r + 1$.
(b) P is partitioned into disjoint sets Q_v and F_v.
(c) Q_v has at least kr points, for all $v \in V$.
(d) F_v is contained in $\mathcal{B}(v, 2r)$, for all $v \in V$.

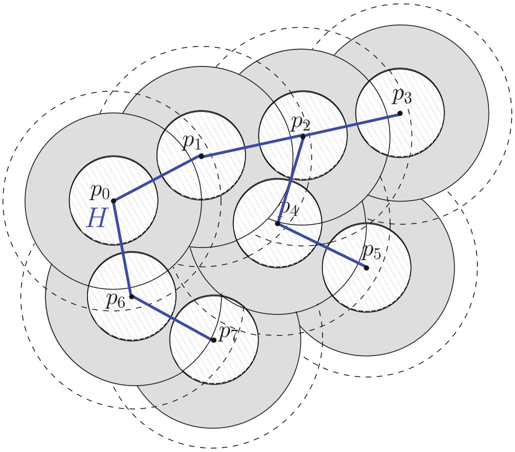

Fig. 1. A partitioning obtained by Algorithm 1. The induced graph H is shown by dark edges.

Proof.

(a) The graph H computed by the algorithm is obviously connected, as each new vertex created by calling PARTITION in line 4 is connected in line 5 to a previous vertex of H. Moreover, lines 2 and 3 of the algorithm enforce that any two adjacent vertices in H have distance between $2r$ and $2r + 1$.

(b) The sets F_v are disjoint by their definition. The sets Q_v are also disjoint, because any two vertices in H have distance more than $2r$ by line 2 of the algorithm.

(c) This is a corollary of Lemma 1.

(d) This is clear from lines 2 and 3 of the algorithm. □

We call each set Q_v a *group*, and the points in F_v the *free points* associated to the group Q_v. We call v the *center* of Q_v. Two groups Q_u and Q_v are called *adjacent groups*, if there is an edge (u, v) in the graph H.

Orienting Antennas. Here, we show how to place antennas with angle at least π on a point set P, so that the resulting communication graph becomes k-connected, with a guaranteed stretch factor. In the rest of this section, we describe our method for $\alpha = \pi$. However, the method is clearly valid for any larger angle.

Theorem 1. *Given a k-connected point set P with at least $2k$ points in the plane, we can place antennas with angle π and radius 9 on P, such that the resulting communication network is k-connected.*

Proof. We run Algorithm 1 with $r = 2$ on the point set P to obtain the graph $H = (V, E)$. For each $v \in V$, let Q_v and F_v be the sets defined in Lemma 2. Since $r = 2$, each set Q_v has at least $2k$ points. We partition Q_v by a horizontal line ℓ_v into two equal-size subsets U_v and D_v, each of size at least k, where points in

U_v (resp., in D_v) are all above (resp., below) ℓ_v. (Points on ℓ_v can be placed in either U_v or D_v.) Now, we orient antennas in D_v upward, and antennas in U_v downward. Moreover, we orient antennas in F_v upward if they are below ℓ_v, and downward if they are above or on ℓ_v (see Fig. 2).

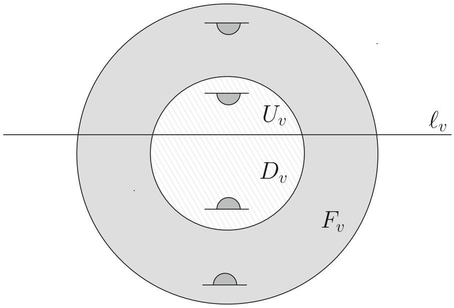

Fig. 2. The orientation of antennas with angle π in $Q_v \cup F_v$.

Let G_π be the communication graph obtained by the above orientation, where the radius of each antenna is set to $4r+1 = 9$. Since each node in D_v has distance at most $2r$ to any node in U_v, Q_v forms a complete bipartite graph, with each part having size at least k, and hence, it is k-connected. Now, we show that the graph on $Q = \cup Q_v$ is k-connected. Note that the distance between the centers of any two adjacent groups Q_u and Q_v is at most $2r + 1$, and the farthest points in the groups have distance at most $4r + 1$. By setting the radius of antennas to $4r + 1$, either all members of D_u connect to all members of U_v, or all members of U_u connect to all members of D_v. So there is a matching of size k between any two adjacent groups, and hence, Q is k-connected by Fact 1. Since F_v is contained in $\mathcal{B}(v, 2r)$, the farthest points in $Q_v \cup F_v$ are at distance $4r$, and hence, each node in F_v connects to at least k nodes in Q_v. Therefore, the whole communication graph is k-connected by Fact 2. □

Theorem 2. *Given a k-connected point set P with at least $2k$ points in the plane, we can place antennas with angle π and radius 13 on P, such that the resulting communication network is a k-fault-tolerant 3-spanner.*

Proof. We use the same orientation described in the proof of Theorem 1. Now, we show that by setting radius of antennas to $6r + 1 = 13$, the resulting graph G_π is a k-fault-tolerant 3-spanner. Fix a set $S \subseteq P$ with $|S| < k$. We show that for any edge $(p, q) \in \mathrm{UDG}(P) \setminus S$, there is a path between p and q in $G_\pi \setminus S$ of length at most 3. Let $T_v = Q_v \cup F_v$. Suppose $p \in T_u$ and $q \in T_v$. Assume w.l.o.g. that ℓ_u is below or equal to ℓ_v. Since $\|pq\| \leq 1$, the centers of Q_u and Q_v are at most $4r + 1$ apart. Therefore, by setting the radius to $6r + 1$, we have a matching of size k between D_u and U_v in G_π. We distinguish the following four cases based on the order of points and lines on the y-axis:

– $p \leq \ell_u$ and $q \leq \ell_v$. Since $|S| < k$, there is a vertex $w \in U_v \setminus S$ such that p and q are both connected to w. Therefore, $\delta_G(p, q) = 2$ in this case.
– $p \leq \ell_u$ and $q > \ell_v$. Since $|S| < k$, there is an edge $(w, x) \in (D_u \setminus S, U_v \setminus S)$. Now, the path $\langle p, x, w, q \rangle$ is a path of length 3 in G.
– $p > \ell_u$ and $q > \ell_v$. Since $|S| < k$, there is a vertex $w \in D_u \setminus S$ such that p and q are both connected to w. Therefore, $\delta_G(p, q) = 2$ in this case.
– $p > \ell_u$ and $q < \ell_v$. This case is analogous to the second case. □

4 Antennas with $\pi/2 < \alpha < \pi$

We now pay our attention to a more challenging case where the goal is to orient the antennas with angle $\pi/2 < \alpha < \pi$ on a point set P, so that the resulting communication graph becomes k-connected. Let $\delta = \lceil 4/|\cos\alpha| \rceil$. We distinguish two cases based on the distribution of P on the plane. P is called *α-sparse* if the diameter of P (i.e. the distance of the farthest pair of points in P) is at least δ. Otherwise, P is called *α-dense.*

Lemma 3. *If P is α-sparse, then the diameter of $P \cap \mathcal{B}(p, \delta + 3)$ is at least δ, for any $p \in P$.*

Proof. Let (q, q') be the farthest pair of points in P. If both q and q' are contained in $\mathcal{B}(p, \delta + 3)$, we are done. Otherwise, at least one of q and q' (say q) is outside $\mathcal{B}(p, \delta + 3)$. Since UDG($P$) is connected, $\mathcal{A}(p, \delta + 1)$ must contain some point t of P. Since t is inside $\mathcal{B}(p, \delta + 1)$ and $\|tp\| > \delta$, the diameter of $P \cap \mathcal{B}(p, \delta + 3)$ is at least δ. □

Algorithm Sketch. We first sketch the whole algorithm, and then go into details of each part. The algorithm is almost similar to the one given in the previous section for $\alpha = \pi$. We run Algorithm 1 with $r = \delta + 3$ on the point set P to obtain the graph $H = (V, E)$. We then add edges to H to make any two vertices of H whose distance is at most $4r + 1$ adjacent. For each $v \in V$, let Q_v and F_v be the sets defined in Lemma 2. We orient antennas in $Q_v \cup F_v$ such that the resulting graph is k-connected. We then make the radius of the antennas large enough, so that for any two adjacent groups Q_u and Q_v, their union (and consequently $Q = \cup Q_v$) becomes k-connected.

Observation 1. *If P is α-dense, $H = (V, E)$ is a single vertex.*

We start explaining how to make each Q_v k-connected. We define *α-cone* to be a cone with angle $2\alpha - \pi$. Let $\sigma(c)$ be an α-cone with apex c and let $\bar{\sigma}(c)$ be the reflection of $\sigma(c)$ about c. Our algorithm relies on the following lemma.

Lemma 4.

– *If the diameter of Q_v is at least δ, then there is an α-cone $\sigma(c)$ for some point c on the plane such that both $\sigma(c)$ and $\bar{\sigma}(c)$ contain at least $2k$ points of Q_v.*

– *If the diameter of Q_v is less than δ but Q_v contains at least $8k \cdot \pi/(2\alpha - \pi)$ points, then there is an α-cone $\sigma(c)$ for some point c on the plane such that both $\sigma(c)$ and $\bar{\sigma}(c)$ contain at least $2k$ points of Q_v.*

The proof of this lemma is omitted in this version due to lack of space.

We recall that if P is α-sparse, the diameter of each set Q_v is at least δ. If P is α-dense, we only have one set Q_v and then we only need the extra assumption that P contains at least $8k \cdot \pi/(2\alpha - \pi)$ points, in order to use the lemma in our algorithm.

Orienting $Q_v \cup F_v$. Let $\sigma(c)$ be the α-cone obtained in Lemma 4. Let ℓ_1 and ℓ_2 be the lines passing through the sides of $\sigma(c)$ (and $\bar{\sigma}(c)$ as well), and let ℓ be the bisector of the angle $2\pi - 2\alpha$ whose sides are ℓ_1 and ℓ_2 (see Fig. 3 to get more intuition). We define and depict four types of orienting antennas with angle α in Fig. 3 naming O_1, O_2, O_3 and O_4. In each type, each side is parallel to one of the lines ℓ_1, ℓ_2, and ℓ.

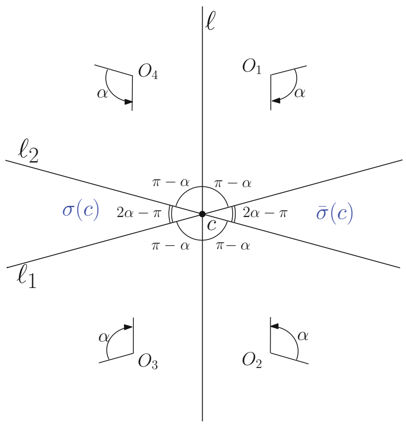

Fig. 3. Cones $\sigma(c)$ and $\bar{\sigma}(c)$, and four orientations with angle α.

Backbone Antennas. We select $2k$ point of $Q_v \cap \sigma(c)$ and arbitrarily partition them into two sets D_v and U_v of size k. Similarly, we select $2k$ point of $Q_v \cap \bar{\sigma}(c)$ and arbitrarily partition them into two sets $\bar{D}_v$ and $\bar{U}_v$ of size k. We use types O_1, O_2, O_3, and O_4 for orienting antennas in $D_v, U_v, \bar{D}_v$, and $\bar{U}_v$, respectively. We call each of these four sets a backbone set. Regardless of the antennas radii, this orientation holds the following properties:

– Each antenna in $D_v \cup U_v$ covers each antenna in $\bar{D}_v \cup \bar{U}_v$ and vice versa.
– Each point in the plane is covered by all antennas in one of the backbone sets.

To orient antenna p in $Q_v \cup F_v$ other than backbone antennas, we detect which backbone set covers p (i.e. p is visible from all antennas in the backbone set). Let O_i be the orientation type used to orient the backbone set. We orient p with type $\bar{O}_i$ where $\bar{O}_i$ is the reflection of O_i about its apex. Figure 4 depicts how to orient antennas depending on their subdivisions induced by ℓ_1, ℓ_2, and ℓ.

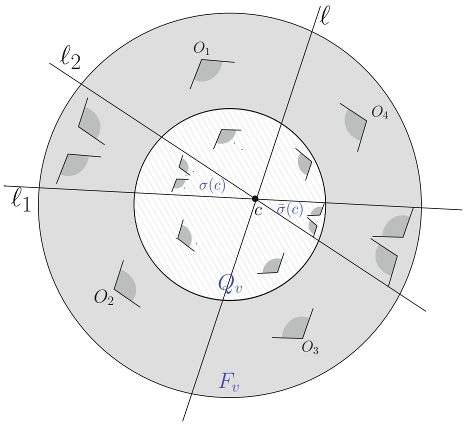

Fig. 4. The orientation of antennas with angle $\pi/2 < \alpha < \pi$ in $Q_v \cup F_v$

Radius. If P is α-dense, we set the radius to be δ as the distance of any two antennas is at most δ. For the α-sparse set P, we need that any two visible backbone antennas u' and v' from two adjacent groups Q_u and Q_v cover each other. Since their distance is at most $\|u'u\| + \|uv\| + \|vv'\| \leq r + 4r + 1 + r \leq 6(\delta + 3) + 1$, we set the radius to be $6\delta + 19$.

***k*-Connectivity.** For any $v \in V$, the induced graph over $(D_v \cup U_v, \bar{D}_v \cup \bar{U}_v)$ is a bipartite complete graph. Moreover, any antenna in $Q_v \cup F_v$ other than the backbone antennas has a direct connection with at least k backbone antennas. All these simply imply that the induced graph over $Q_v \cup F_v$ is k-connected.

Lemma 5. *Suppose $p, q \in Q_v \cup F_v$ and q is a backbone antenna. p and q are in connection with each other via at most three links, even if at most $k-1$ antennas are destroyed.*

Proof. Assume w.l.o.g. that $q \in D_v$. We know p is visible from all members of one backbone set. This backbone set can be either $D_v, U_v, \bar{D}_v$, or $\bar{U}_v$. If this backbone set is either $D_v, \bar{D}_v$ or $\bar{U}_v$, we reach q from p with at most two links.

Otherwise, with 3 links we can get q from p. Since each backbone set has k members and any member of $D_v \cup U_v$ is visible to $\bar{D}_v \cup \bar{U}_v$ and vice versa, the proof works even if at most $k-1$ antennas are destroyed. □

Graph $H = (V, E)$ has only one vertex if P is α-dense. Therefore, using Lemma 5 we can simply show any two points are in connection with other via at most 4 links even if $k-1$ antennas are destroyed. Note that any antenna is either a backbone antenna or directly connected to a backbone antenna. Next we assume P is α-sparse.

Here, we need to show the connection of two adjacent groups Q_v and Q_u remain safe even if $k-1$ antennas are destroyed. We partition the backbone antennas in Q_v (similarly in Q_u) into k sets S_v^i ($i = 1, \ldots, k$) of size 4, each containing one antenna from the sets D_v, U_v, $\bar{D}_v$, and $\bar{U}_v$. We know each point in the plane is visible from one member of S_v^i, and moreover, two sets S_v^i and S_u^i can be separated by a line. This together with the following proposition implies that there are two backbone antennas $p \in S_v^i$ and $q \in S_u^i$ which are visible to each other, and hence, with the radius specified for antennas they are in the coverage area of each other.

Proposition 1 ([2]). *Let A and B be two sets containing 4 antennas with angle at least $\pi/2$. Suppose both A and B cover the entire plane regardless of the antennas radius. If there exists a line ℓ that separates A and B, then by setting the radius unbounded, the network induced by $A \cup B$ is connected.*

The above discussion shows that there are at least k distinct links between the backbone antennas of two adjacent groups Q_v and Q_u. Therefore, even if $k-1$ antennas are destroyed, the connection between Q_v and Q_u remains safe. This together with Lemma 5 implies that for any two antennas $p \in Q_v \cup F_v$ and $q \in Q_u \cup F_u$, there is a connection via at most 7 links.

Stretch Factor. Let p and q be two arbitrary points in P, and let $x_0 = p, x_1, \ldots, x_t = q$ be the shortest link distance between p and q in $\mathrm{UDG}(P) \setminus S$, where S is the fault set with size at most $k-1$. Since $\|x_i x_{i+1}\| \leq 1$, either there exists $v \in V$ such that $x_i, x_{i+1} \in Q_v \cup F_v$, or there exist two adjacent $u, v \in V$ such that $x_i \in Q_v \cup F_v$ and $x_{i+1} \in Q_u \cup F_u$. This shows that in the communication graph obtained by our algorithm, each link (x_i, x_{i+1}) either exist or is replaced by a path of length at most 4 in the α-dense set P, and a path of length at most 7 in the α-sparse set P. Therefore, our resulting graph is a 4-spanner and a 7-spanner for the α-dense set P and the α-sparse set P, respectively.

Putting all these together, we get the main theorem of this section.

Theorem 3. *Suppose P is a k-connected point set in the plane, and α is a given angle in the range $(\pi/2, \pi)$. Let $\delta = \lceil 4/|\cos\alpha| \rceil$. Then, the followings hold:*

- *If P is α-sparse, we can place antennas with angle α and radius $6\delta+19$ on P, such that the resulting communication network is a k-fault-tolerant 7-spanner.*

– *If P is α-dense and contains at least $8k\cdot\pi/(2\alpha-\pi)$ points, we can place antennas with angle α and radius δ on P, such that the resulting communication network is a k-fault-tolerant 4-spanner.*

5 Concluding Remarks

In this paper, we studied the problem of constructing fault-tolerant spanners in networks with symmetric directional antennas, and presented the first algorithms for placing antennas with angles $\alpha > \pi/2$, so that the resulting communication graph is a k-fault-tolerant t-spanner, for small stretch factors $t \leq 7$.

Throughout this paper, we assumed that UDG(P) is k-connected. This assumption can be relaxed to the connectivity of UDG(P) at the expense of increasing the radius and stretch factor. If we replace the k-connectivity with a 1-connectivity assumption, the radius of antennas implied by Lemma 1 is multiplied by k, and hence, the radius and stretch factor of our constructed network is increased by a factor of k. For example, on a point set whose UDG is connected, our algorithm constructs a k-fault-tolerant spanner with radius $13k$ and stretch factor $3k$. A natural open problem is to find fault-tolerant spanners with smaller radius and/or stretch factors. The case $\pi/3 \leq \alpha \leq \pi/2$ is also open for further investigation.

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